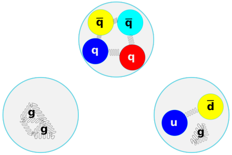
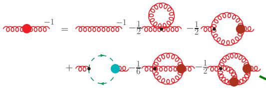
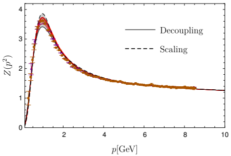
From correlation functions to glueballs

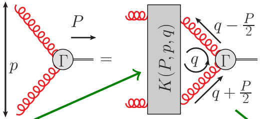
Markus Q. Huber

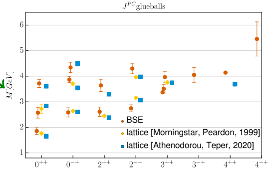
Institute of Theoretical Physics, Giessen University











HFHF Kolloquium

June 6, 2024

Giessen, Germany

JUSTUS-LIEBIG-UNIVERSITÄT GIESSEN

Deutsche Forschungsgemeinschaft

Bound states

Under some attractive force, constituents form a “state” that behaves like a single object (under certain conditions).

Constituents bound by some force.

- Localized
- Attractive force
- Behaves as a single object (under certain conditions)
- Discrete spectrum (as opposed to free constituents)
- 2 or more constituents

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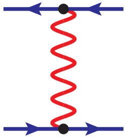

2 fermions in QED:

- Example: Hydrogen atom
- one-photon exchange
- Coulomb potential $\propto 1/r$
- spin-orbit coupling: fine splitting
- spin-spin coupling: hyperfine splitting

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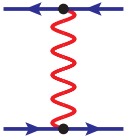

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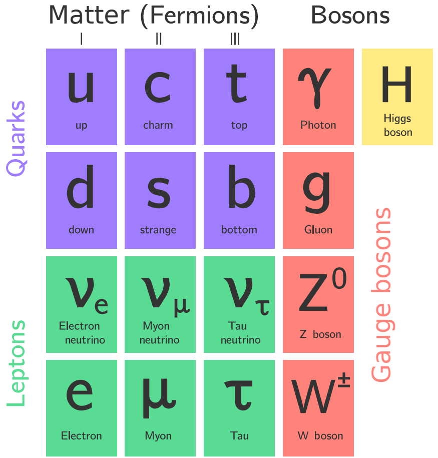
Caveat: Many QCD bound states are not stable and decay. $\rightarrow$ Resonances.

Bound states in the standard model



Standard model: matter + exchange particles, electroweak and strong forces

(Elementary) particles form bound states:

- QED: proton + electron $\rightarrow$ hydrogen atom

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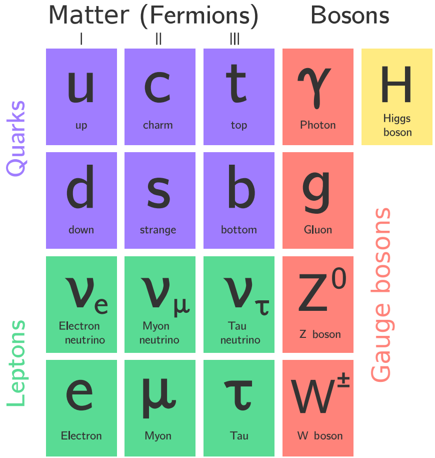

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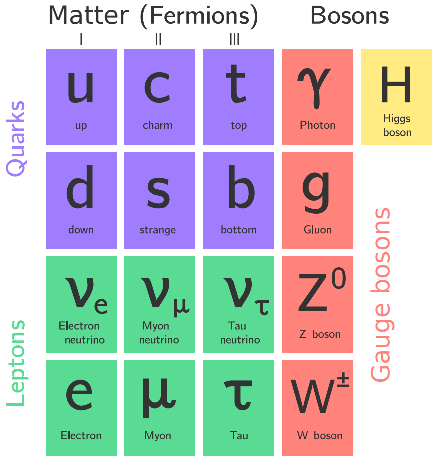

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- Standard model: FMS mechanism

Mesons and baryons

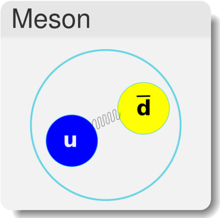
Proton, neutron (1932), pion (1947)

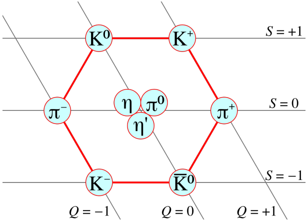
Explosion of discoveries of new hadrons in 50s, 60s → “particle zoo”

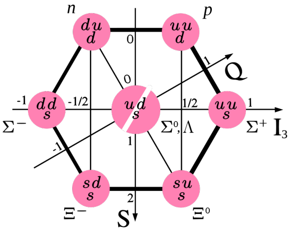
Pauli: “Had I foreseen that, I would have gone into botany.“

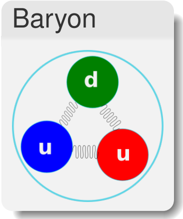
Quark model:

Classification in multiplets of mesons (quark+antiquark) or baryons (3 quarks) (”eightfold way“):



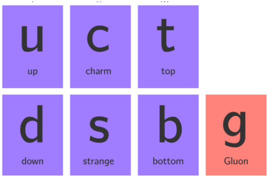






Quantum chromodynamics (QCD)

- (Elementary) theory
- Non-Abelian gauge theory
- Built on prototype of QED



Quarks: matter fields

Gluons: interaction fields

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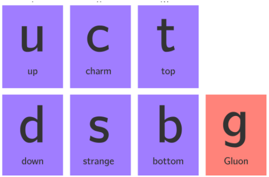

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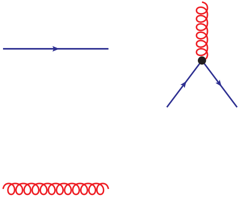

$$\mathcal{L}_{\text{QED}} = \bar{\psi}\,(-\not{D} + m)\psi + \frac{1}{2}\,\text{Tr}\{F_{\mu\nu}F^{\mu\nu}\}$$

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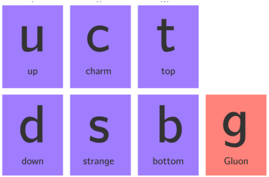

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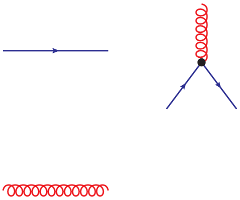

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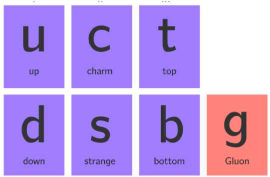

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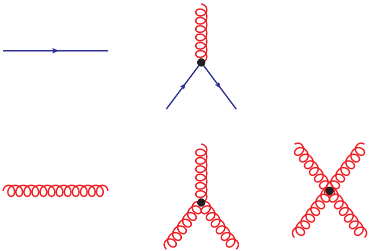

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Beyond mesons and baryons

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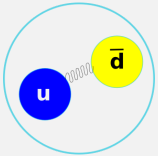
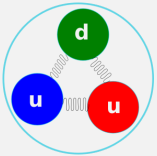


Baryon



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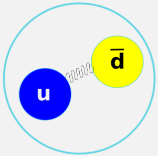

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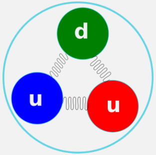
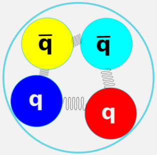


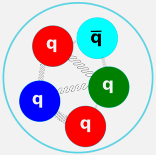
- More than 3 quarks?
  → Tetraquarks, pentaquarks

Exotics:

Tetraquark



Pentaquark



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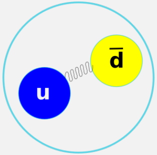

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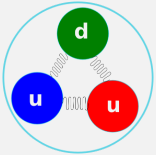

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- Bound states of gluons?
  → Glueballs

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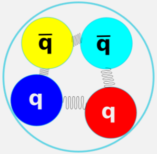

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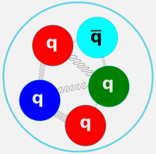

Glueball

Beyond mesons and baryons

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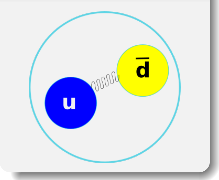

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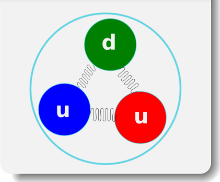

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- Quarks and gluons?
  $\rightarrow$ Hybrids

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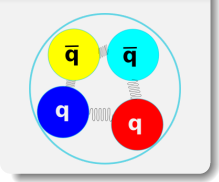

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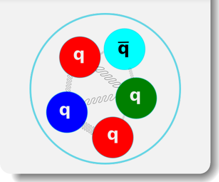
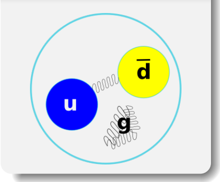


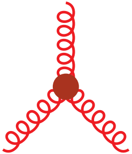
Glueball

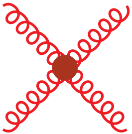
Hybrid



Glueballs

Non-Abelian nature of QCD $\rightarrow$ self-interaction of force fields.





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Glueballs from gauge inv. operators, e.g., $F_{\mu\nu}F^{\mu\nu}$.

$\rightarrow$ Mixing of operators with equal quantum numbers.

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Experiment:
Production in glue-rich environments, e.g., $p\bar{p}$ annihilation (PANDA), pomeron exchange in pp (central exclusive production), radiative J/ψ decays

Reviews on glueballs: [Klempt, Zaitsev, Phys.Rept.454 (2007); Mathieu, Kochelev, Vento, Int.J.Mod.Phys.18 (2009); Crede, Meyer, Prog.Part.Nucl.Phys.63 (2009); Ochs, J.Phys.G40 (2013); Llanes-Estrada, EPJST 230 (2021); Vadacchino, 2305.04869]

Glueball predictions

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- Functional studies $\rightarrow$ This talk

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→ Consensus: $\mathbf{0^{++}}$ is lightest state; (next 0^{-+}, 2^{++}, 0^{++*})

Glueball calculations: Lattice

Nonperturbative first-principles method: discretized space-time, continuum limit, computationally expensive

- [Gregory et al., JHEP10 (2012)]
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- [Vadacchino, Lattice2022, 2305.04869]
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Challenges [Gregory et al., JHEP10 (2012)]:

- Much higher statistics required (poor signal-to-noise ratio)
- Continuum extrapolation and inclusion of fermionic operators still to be done
- $m_\pi = 360\,\text{MeV}$
- Mixing with $\bar{q}q$ challenging
- Small unquenching effects found

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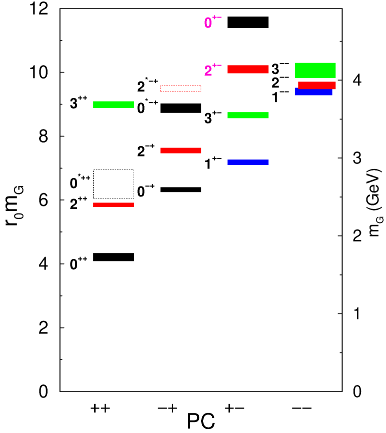
No quantitative results yet.

Glueballs: Quenched lattice calculations

Make quarks infinitely heavy
$\rightarrow$ no quark dynamics
$\rightarrow$ no mixing
$\rightarrow$ pure gauge theory, Yang-Mills theory
$\rightarrow$ "Pure" glueballs

- [Morningstar, Peardon, Phys. Rev. D60 (1999)]: standard reference
- [Athenodorou, Teper, JHEP11 (2020)]: improved statistics, more states

Quantitative results $\rightarrow$ Benchmark



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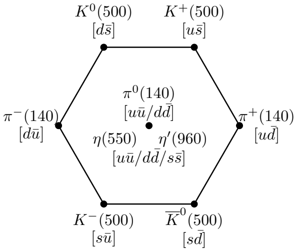
Scalar sector

$J^{PC} = 0^{++} \rightarrow q\bar{q}$ mesons, tetraquarks and glueballs

$m_u \sim m_d < m_s$



Pseudoscalar:
mass hierarchy ✓

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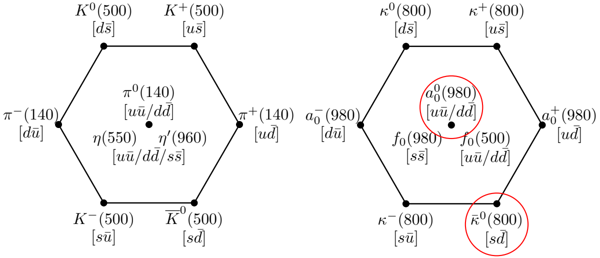

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Scalar:
Mass hierarchy?

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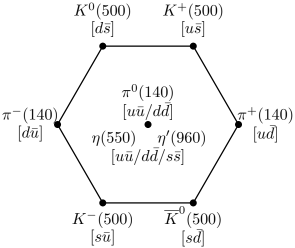

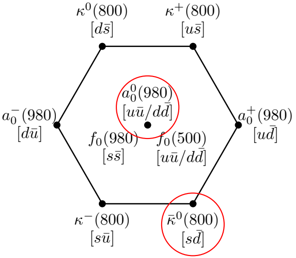

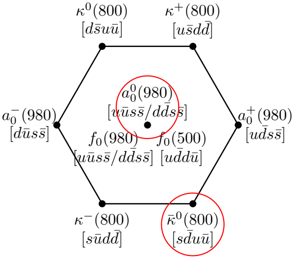

Functional review:
[Eichmann, Fischer, Santowsky, Wallbott, Few-Body Syst.61 (2020)]

Pseudoscalar:
mass hierarchy ✓

Scalar:
Mass hierarchy?

Scalar:
Inverted mass hierarchy if tetraquarks
[Jaffe, Phys. Rev. D 15 (1977)]

Scalar sector

$J^{PC} = 0^{++} \rightarrow q\bar{q}$ mesons, tetraquarks and glueballs

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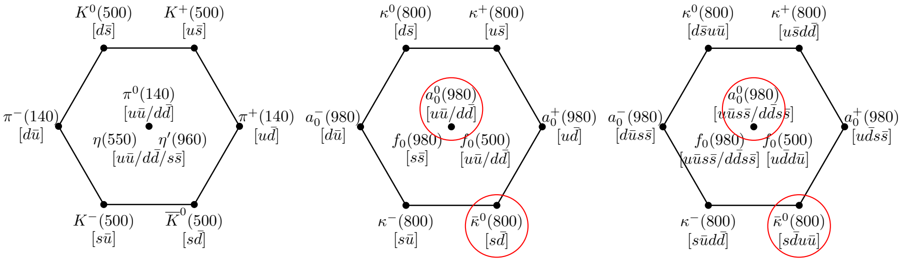

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Scalar: Mass hierarchy?

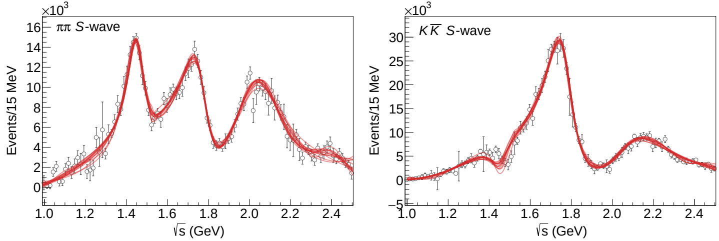
Scalar: Inverted mass hierarchy if tetraquarks [Jaffe, Phys. Rev. D 15 (1977)]

Scalar glueballs: $Q = 0$, isoscalar

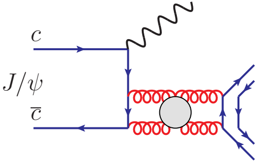
$f_0(500)$	$f_0(1370)$
$f_0(980)$	$f_0(1500)$
	$f_0(1710)$

← glueball candidates ($f_0(1370)$, $f_0(1500)$, $f_0(1710)$)

Glueballs from J/ψ decay



[JPAC Coll., Rodas et al., Eur.Phys.J.C 82 (2022)]



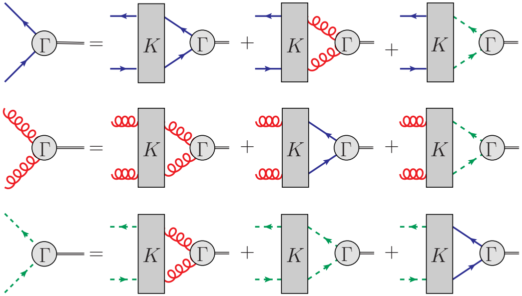
Scalar glueball candidate: Coupled-channel analyses of exp. data (BESIII)

- +add. data, largest overlap with $f_0(1770)$ [Sarantsev, Denisenko, Thoma, Klempt, Phys. Lett. B 816 (2021)]
- largest overlap with $f_0(1710)$ [JPAC Coll., Rodas et al., Eur.Phys.J.C 82 (2022)]

Pseudoscalar glueball candidate:

- $X(2370)$ [Ablikim et al. (BESIII), PRL132 (2024)]

Methods:
Bound state equations



Elements of a BSE

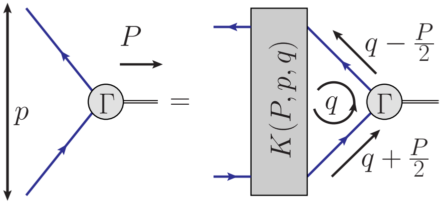
$$\Gamma = K\, G_0\, \Gamma$$

Input:

- Propagators G_0
- Kernel K

Output:

- Mass M: $M^2 = -P^2$
- Bethe-Salpeter amplitudes Γ



(quark-antiquark state)

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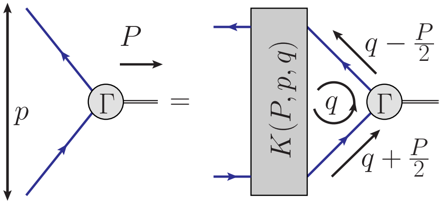


(quark-antiquark state)

Approximations: bottom-up $\longleftrightarrow$ top-down

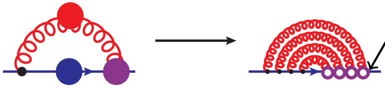
Bottom-up approximation: Rainbow

Need the gluon propagator ($Z(k^2)$) and the quark-gluon vertex ($h_i(k;p,q)$).

Bottom-up approximation: Rainbow

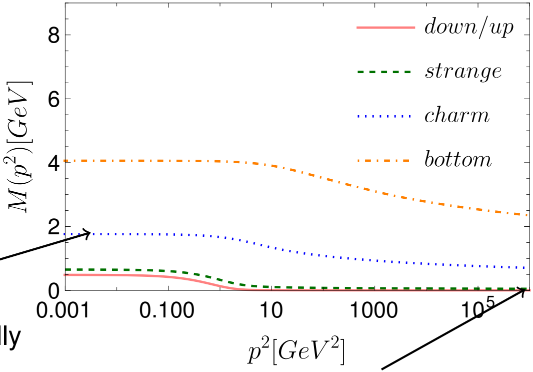
Need the gluon propagator ($\boldsymbol{Z(k^2)}$) and the quark-gluon vertex ($\boldsymbol{h_i(k;p,q)}$).

- $\boldsymbol{\Gamma^{a,\nu}(k;p,q)} \propto \gamma^\nu h_1(k;p,q)$
- $\frac{g^2}{4\pi} Z(k^2) h_1(k;p,q) \propto \boldsymbol{\alpha(k^2)}$ ← 'model interaction'



Iteration → only 'rainbow-like' diagrams

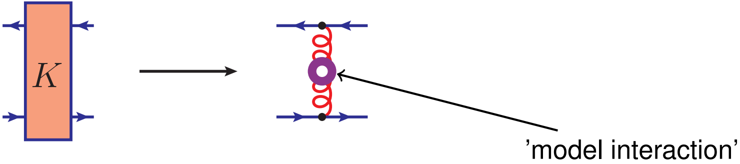
'constituent quark mass' from dynamically broken chiral symmetry → QCD



'current quark mass', QCD parameter

Bottom-up approximation: Ladder

Scattering kernel: “all interactions which are two-particle irreducible with respect to two horizontal quark lines”



Same model $\rightarrow$ Rainbow+ladder approximation respects chiral symmetry (axial-vector WTI).

Functional spectrum calculations: Bottom-up

Models, qualitative insight, quantitative results for some cases

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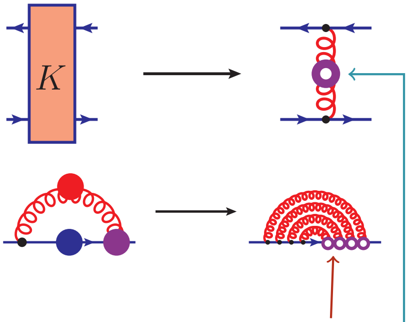


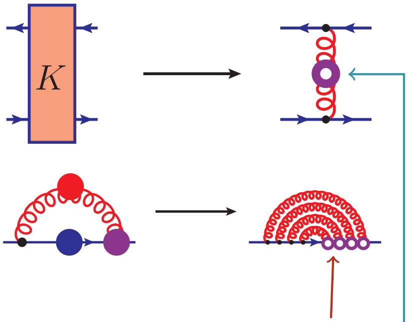
restricted structure of equations ($\Gamma_\mu \to \gamma_\mu$)

IR strength + perturbative UV

Workhorse for more than 20 years: Rainbow-ladder truncation with an effective interaction, e.g., Maris-Tandy (or similar).

Functional spectrum calculations: Bottom-up

Models, qualitative insight, quantitative results for some cases



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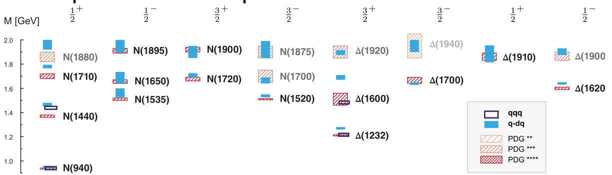
Example: Nucleon spectrum

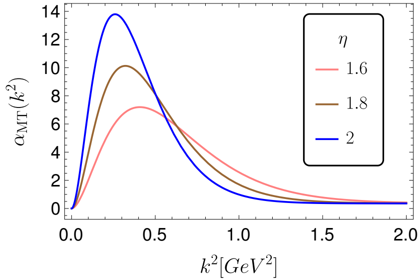


[Eichmann, Sanchis-Alepuz, Williams, Alkofer, Fischer, Prog.Part.Nucl.Phys. 91 (2016); Eichmann, Few Body Syst. 63 (2022)]

Example for a model: Maris-Tandy interaction

[Maris, Roberts, Tandy, Phys. Rev. C 56 (1997); Maris, Tandy, Phys. Rev. C 60 (1999)]:

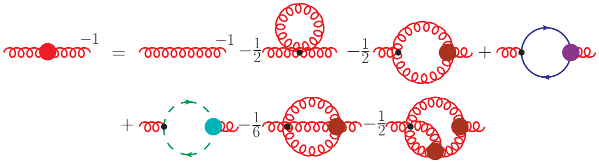
$$\alpha(k^2) = \underbrace{\pi\,\eta^7 \left(\frac{k^2}{\Lambda^2}\right)^2 e^{-\eta^2 \frac{k^2}{\Lambda^2}}}_{\alpha_{\mathrm{IR}}(k^2)} + \alpha_{\mathrm{UV}}(k^2)$$



- Scale Λ from f_π
- Quark masses $m_u = m_d$, m_s from m_π, m_K
- Parameter η: window of small sensitivity (for meson masses and decay constants)
- α_{UV}: Phenomenologically irrelevant, provides correct perturbative running to quark propagator

Functional glueball calculations

Glueballs? Rainbow-ladder?



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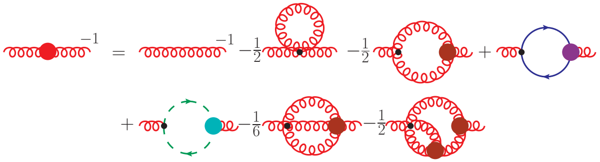

There is no rainbow for gluons!

Functional glueball calculations

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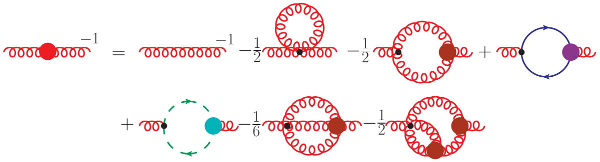

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Model based BSE calculations ($J = 0$):

- [Meyers, Swanson, Phys.Rev.D87 (2013)]
- [Sanchis-Alepuz, Fischer, Kellermann, von Smekal, Phys.Rev.D92, (2015)]
- [Souza et al., Eur.Phys.J.A56 (2020)]
- [Kaptari, Kämpfer, Few Body Syst.61 (2020)]

Functional glueball calculations

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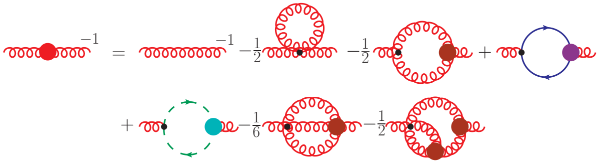

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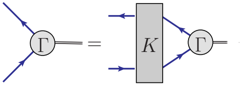
- [Souza et al., Eur.Phys.J.A56 (2020)]
- [Kaptari, Kämpfer, Few Body Syst.61 (2020)]

Alternative: Calculated input [MQH, Phys.Rev.D 101 (2020)]

- $J = 0$: [MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C80 (2020)]
- $J = 0, 2, 3, 4$: [MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C81 (2021)]

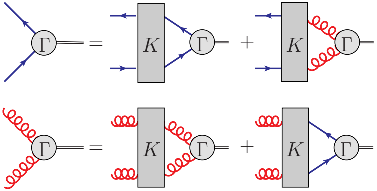
Extreme sensitivity on input!

Bound state equations for QCD



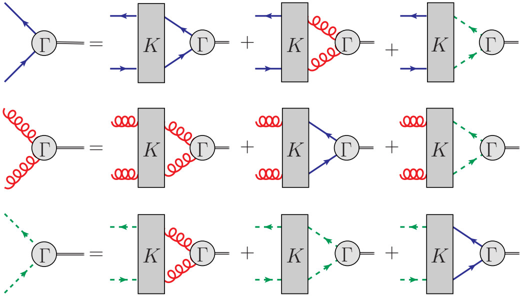
- Require scattering kernel K and propagator.

Bound state equations for QCD



- Require scattering kernels *K* and propagators.
- Quantum numbers determine which amplitudes Γ couple.

Bound state equations for QCD



- Require scattering kernels K and propagators.
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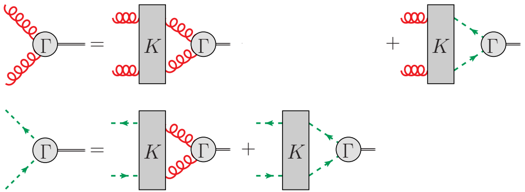
- Ghosts from gauge fixing

One framework

- Natural description of mixing.
- Similar equations for hadrons with more than two constituents

Bound state equations for QCD

Focus on pure glueballs.



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One framework

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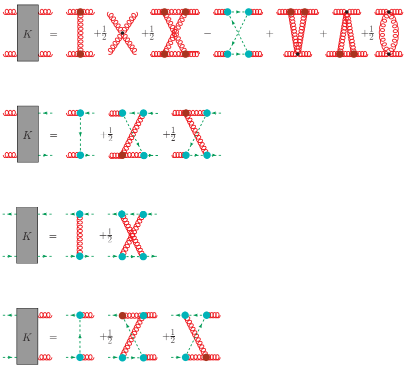
- Similar equations for hadrons with more than two constituents

Kernels

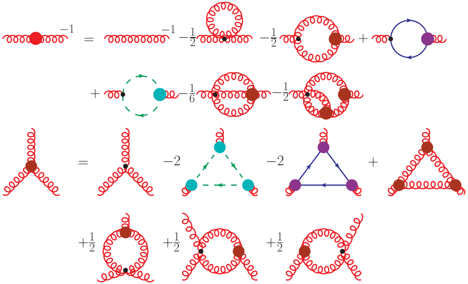
Systematic derivation from 3PI effective action: [Berges, Phys. Rev. D 70 (2004); Carrington, Gao, Phys. Rev. D 83 (2011)]

Self-consistent treatment of 3-point functions requires 3-loop expansion.



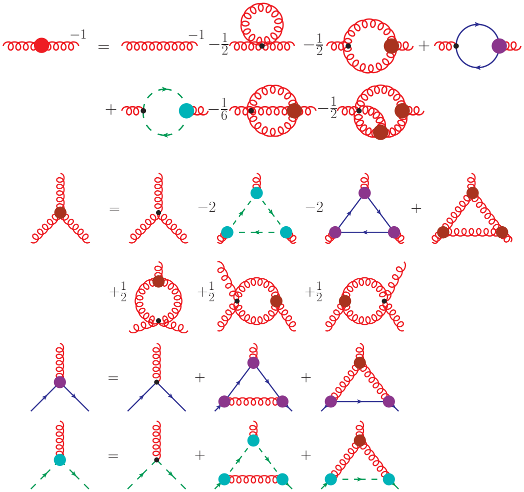
[Fukuda, Prog. Theor. Phys 78 (1987); McKay, Munczek, Phys. Rev. D 40 (1989); Sanchis-Alepuz, Williams, J. Phys: Conf. Ser. 631 (2015); MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C80 (2020)]

Correlation functions



Equations of motion

From 3-loop 3PI effective action



- Conceptual and technical challenges: nonperturbative renormalization, two-loop diagrams, convergence, size of kernels, . . .
- Self-contained: Only parameters are the strong coupling and the quark masses!
- Long way, e.g., ghost-gluon vertex, three-gluon vertex, four-gluon vertex, . . .
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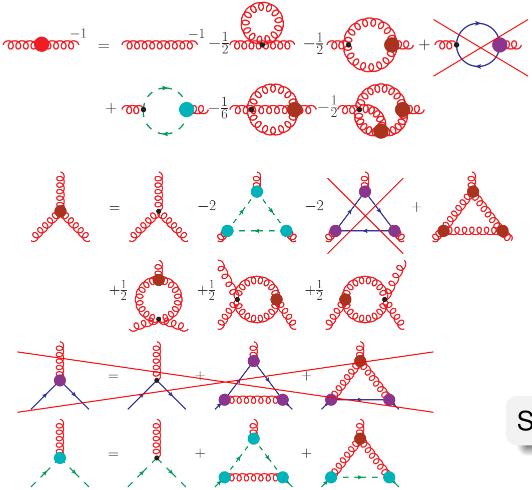

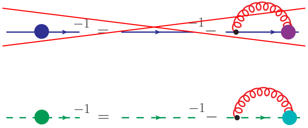

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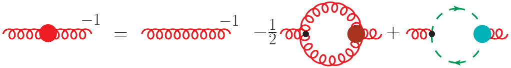
- → MQH, Phys.Rev.D 101 (2020)

Start with pure gauge theory.

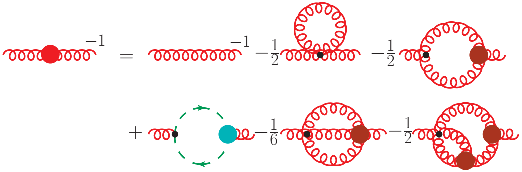
Equations of motion

Early attempts very simple, e.g., gluon propagator with one diagram only [Mandelstam, Phys. Rev. D20 (1979)].

From [von Smekal, Alkofer, Hauck, PRL79 (1997)]



to



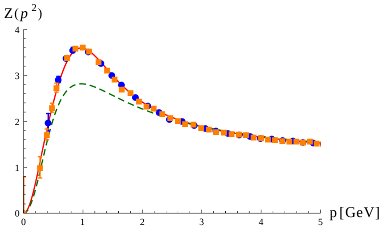
Successive calculation and inclusion of vertices, e.g.,

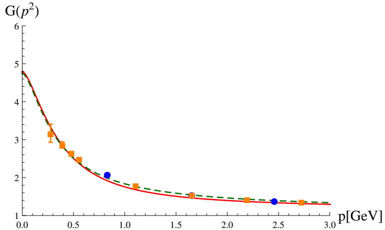
- ghost-gluon vertex [MQH, von Smekal, JHEP 04 (2013)]
- three-gluon vertex [Blum, MQH, Mitter, von Smekal, PRD89 (2014)]
- four-gluon vertex [Cyrol, MQH, von Smekal, EOJC 75 (2015)]

$\longrightarrow$ [Review: MQH, Phys.Rept. 879 (2020)]

The importance of self-consistency

Propagators from modeled three-gluon vertex [MQH, von Smekal, JHEP 04 (2013)]:
Optimize parameters so that propagators match lattice results.





Agreement with lattice is not sufficient!

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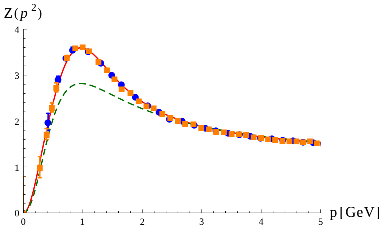

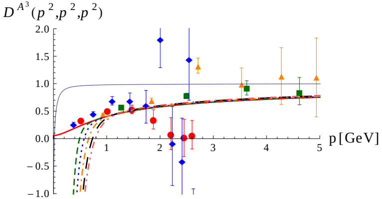

Agreement with lattice is not sufficient!

- Vertex couplings show violations of gauge invariance.
- Model optimized for one quantity. → Restricted applicability to other quantities, e.g., glueballs.

Interlude: Derivation of equations

Automatized derivation with *DoFun*: **D**erivation **o**f **fun**ctional equations

[Alkofer, MQH, Schwenzer, ’08; MQH, Braun, ’11; MQH, Cyrol, Pawlowski, ’19]

Interlude: Derivation of equations

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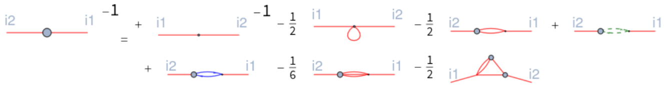
[Alkofer, MQH, Schwenzer, '08; MQH, Braun, '11; MQH, Cyrol, Pawlowski, '19]

→ `https://github.com/markusqh/DoFun/`



Works in two steps:

- Symbolic derivation (no Feynman rules, just types of fields)
- Algebraic: Plug in Feynman rules

doDSE

doDSE[*ac*, *flis*, [*opts*]] derives the DSE from the action *ac* for the fields contained in *flis*.

doDSE[*ac*, *flis*, *props*, [*opts*]] derives the DSE only with propagators contained in prop

doDSE[*ac*, *flist*, *vtest*, [*opts*]] derives the DSE only with vertices allowed by *vtest*.
Allowed propagators will be taken from *ac* if the *props* argument is not given.

▾ Details

- The following options can be given:

Landau gauge propagators

Self-contained: Only external input is the coupling! $\rightarrow$ Ab-initio! [MQH, Phys.Rev.D 101 (2020)]

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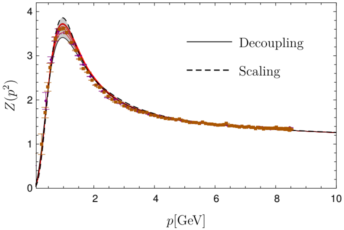
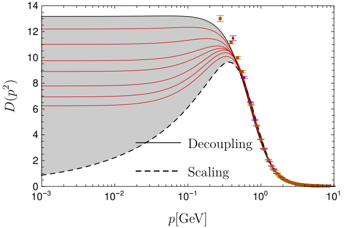
Gluon dressing function:



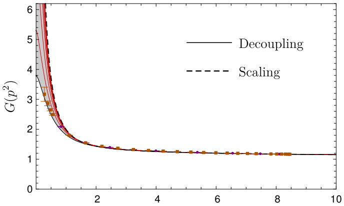
Gluon propagator:



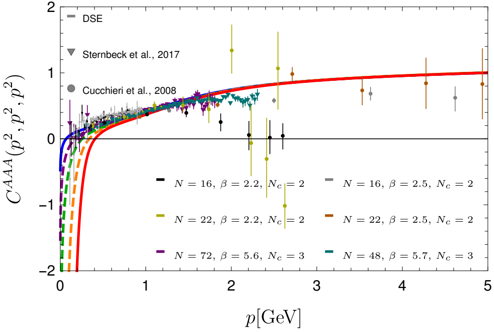
Family of solutions [von Smekal, Alkofer, Hauck, PRL79 (1997); Aguilar, Binosi, Papavassiliou, Phys.Rev.D 78 (2008); Boucaud et al., JHEP06 (2008); Fischer, Maas, Pawlowski, Ann.Phys. 324 (2008); Alkofer, MQH, Schwenzer, Phys. Rev. D 81 (2010)]

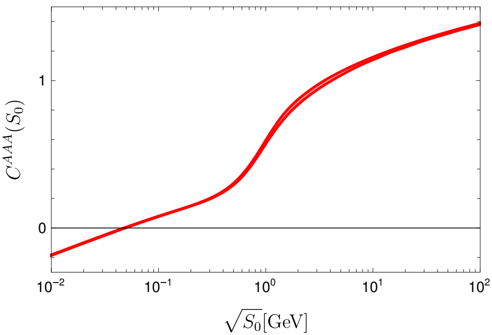
Nonperturbative completions of Landau gauge [Maas, Phys. Lett. B 689 (2010)]?

Ghost dressing function:



Three-gluon vertex I





- Good agreement with lattice results

[Cucchieri, Maas, Mendes, Phys. Rev. D 77 (2008); Sternbeck et al., 1702.00612; MQH, Phys. Rev. D 101 (2020)]

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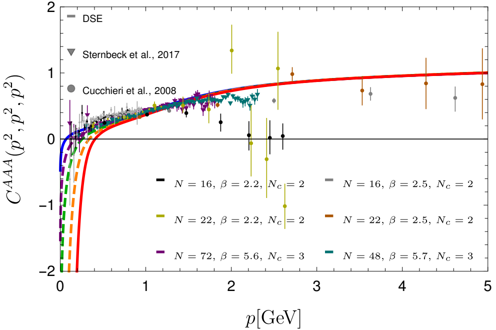
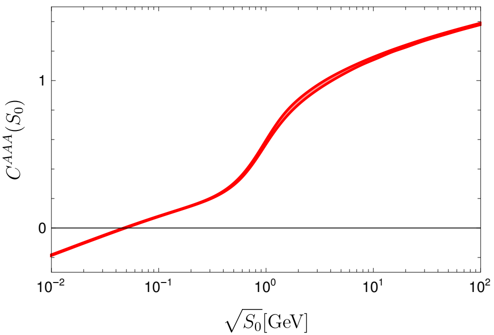
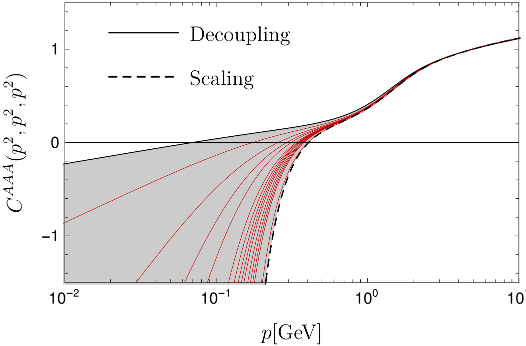


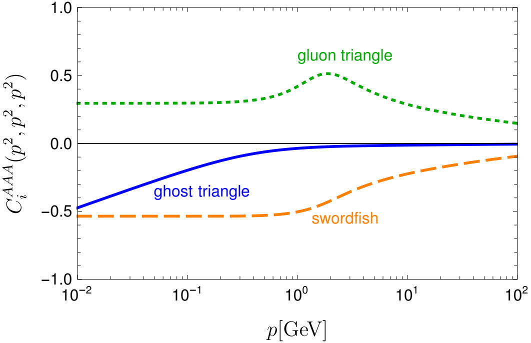


[Cucchieri, Maas, Mendes, Phys. Rev. D 77 (2008); Sternbeck et al., 1702.00612; MQH, Phys. Rev. D 101 (2020)]

- Good agreement with lattice results
- Simple kinematic dependence of three-gluon vertex (only singlet variable of S_3), “planar degeneracy“ confirmed by lattice [Pinto-Gómez et al., Phys.Lett.B838 (2023)]
  First observation: [Eichmann, Williams, Alkofer, Vujinovic, Phys.Rev.D89 (2014)], but already in old data [Blum, Huber, Mitter, von Smekal, Phys.Rev.D89 (2014)] $\rightarrow$ stable property

Three-gluon vertex II





[Cucchieri, Maas, Mendes, Phys. Rev. D 77 (2008); Sternbeck et al., 1702.00612; MQH, Phys. Rev. D 101 (2020)]

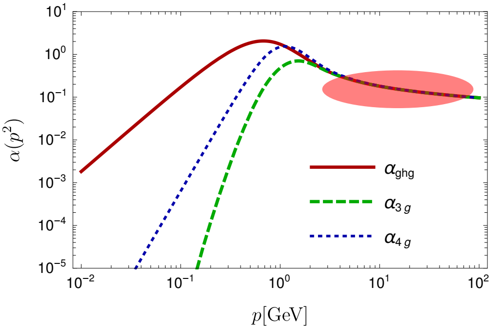
- Family of solutions
- Cancellations between diagrams important

Gauge invariance

[MQH, Phys. Rev. D 101 (2020)]

Couplings can be extracted from each vertex.

- Slavnov-Taylor identities (gauge invariance): Agreement perturbatively (UV) necessary.
  [Cyrol et al., Phys.Rev.D 94 (2016)]
- Difficult to realize: Small deviations → Couplings cross and do not agree.
- Here: Vertex couplings agree down to GeV regime (IR can be different).



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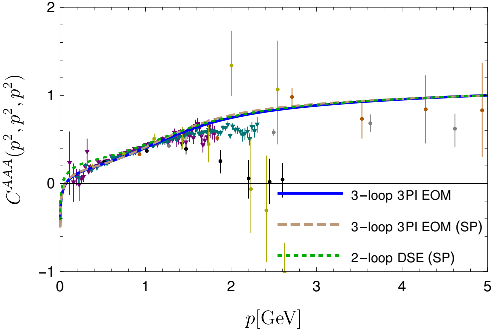
3PI vs. 2-loop DSE:



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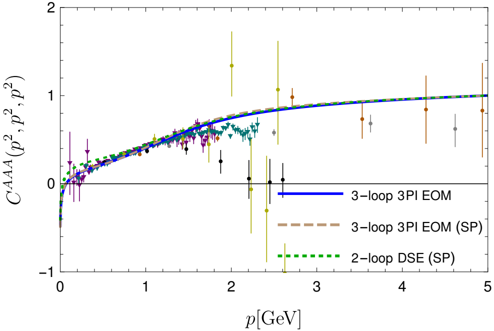
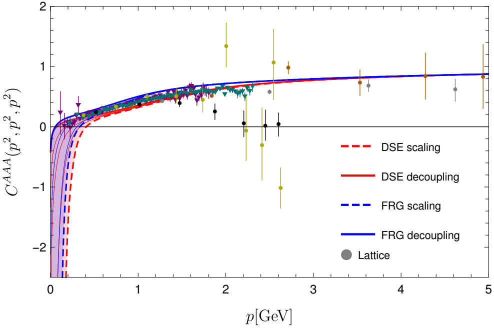


DSE vs. FRG:



[Cucchieri, Maas, Mendes, Phys.Rev.D77 (2008); Sternbeck et al., Proc.Sci. LATTICE2016 (2017); Cyrol et al., Phys.Rev.D 94 (2016); MQH, Phys.Rev.D101 (2020)]

Stability of the solution: Extensions

- Three-gluon vertex: Tree-level dressing dominant, others subleading [Eichmann, Williams, Alkofer, Vujinovic, Phys.Rev.D89 (2014); Pinto-Gómez et al., 2208.01020] ✓

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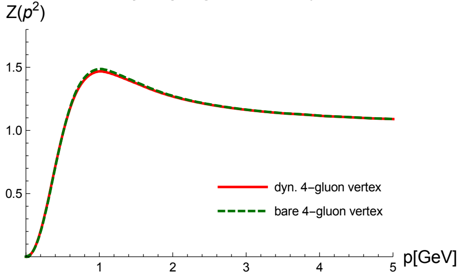
- Four-gluon vertex: Influence on propagators tiny for $d = 3$ [MQH, Phys.Rev.D93 (2016)] ✓



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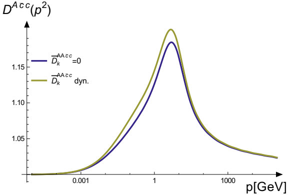
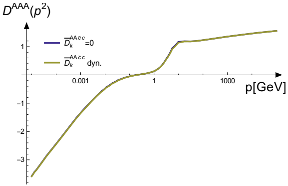
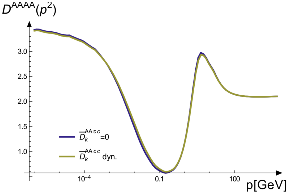
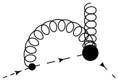
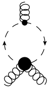
- Four-gluon vertex: Influence on propagators tiny for $d = 3$ [MQH, Phys.Rev.D93 (2016)] ✓
- Two-ghost-two-gluon vertex [MQH, Eur. Phys.J.C77 (2017)]: ✓
  (FRG: [Corell, SciPost Phys. 5 (2018)])

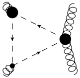






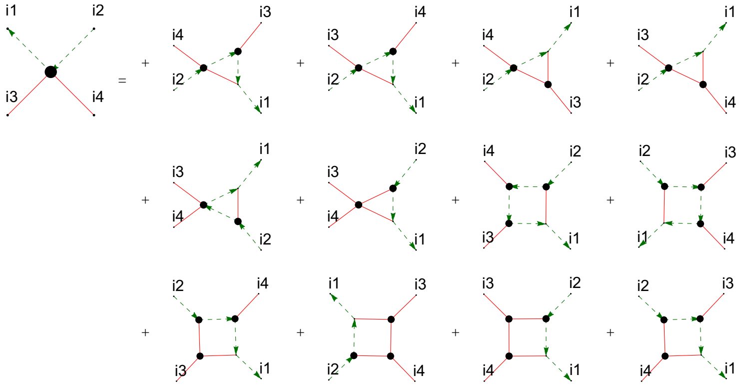






The two-ghost-two-gluon vertex DSE

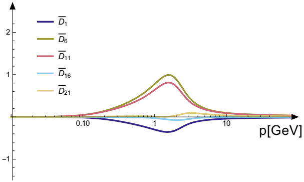
2 DSEs, choose the one with the ghost leg attached to the bare vertex
$\rightarrow$ Truncation discards only one diagram.

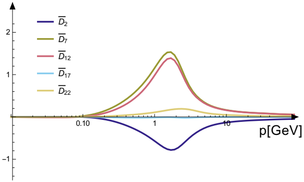


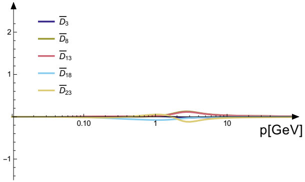
Results for the two-ghost-two-gluon vertex

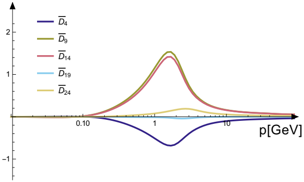
Kinematic approximation: one-momentum configuration

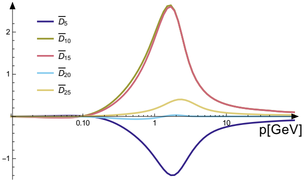
- Dimensionless dressing functions $\overline{D}_k$.
- Each plot one Lorentz tensor.









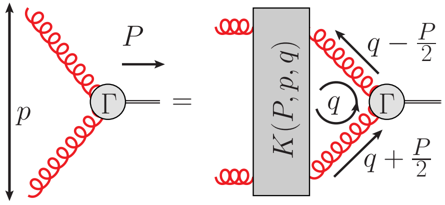


$\rightarrow$ Two classes of dressings: 13 very small, 12 not small

[MQH, Eur.Phys.J.C77 (2017)]

Glueballs

Correlation functions for complex momenta



(pseudoscalar glueball)

$$\boldsymbol{\lambda(P)}\Gamma(P) = \mathcal{K} \cdot \Gamma(P)$$

$\rightarrow$ Eigenvalue problem for $\Gamma(P)$:

1. Solve for $\lambda(P)$.
2. Find P with $\lambda(P) = 1$.
   $\Rightarrow M^2 = -P^2$

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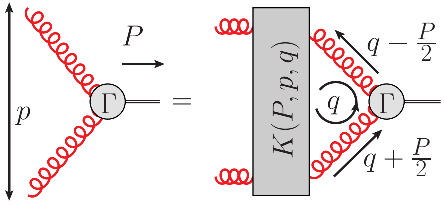

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However:

Propagators are probed at $\left(q \pm \frac{P}{2}\right)^2 = \frac{P^2}{4} + q^2 \pm \sqrt{P^2\, q^2} \cos\theta = -\frac{M^2}{4} + q^2 \pm \boldsymbol{i}\, M \sqrt{q^2} \cos\theta$

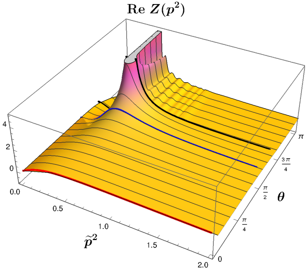
$\rightarrow$ Complex for $P^2 < 0$!

Time-like quantities ($P^2 < 0$) $\rightarrow$ Correlation functions for complex arguments.

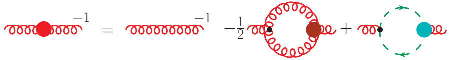
Correlation functions in the complex plane

Gluon

[Fischer, MQH, Phys.Rev.D 102 (2020)]

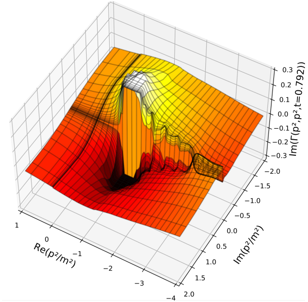


Simpler truncation:



Vertex (scalar toy theory)

[MQH, Kern, Alkofer, Phys.Rev.D107 (2023)]



- Numerically more demanding due to calculations close to cuts.
- Branch cut

Extrapolation of $\lambda(P^2)$

Extrapolation method

- Extrapolation to time-like P^2 using Schlessinger's continued fraction method (proven superior to default Padé approximants) [Schlessinger, Phys.Rev.167 (1968)]
- Average over extrapolations using subsets of points for error estimate

$$f(x) = \frac{f(x_1)}{1 + \frac{a_1(x-x_1)}{1+\frac{a_2(x-x_2)}{1+\frac{a_3(x-x_3)}{\cdots}}}}$$

Coefficients a_i can be determined such that $f(x)$ exact at x_i.

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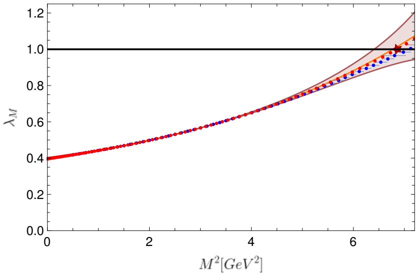
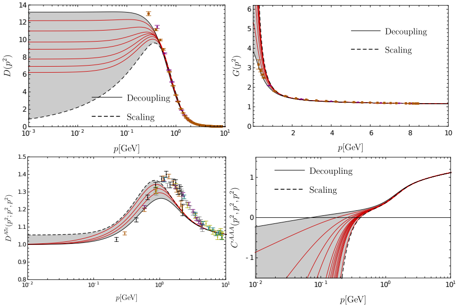
Test extrapolation for solvable system:
Heavy meson [MQH, Sanchis-Alepuz, Fischer, Eur.Phys.J.C 80 (2020)]



Glueball results J=0

Gauge-variant correlation functions:



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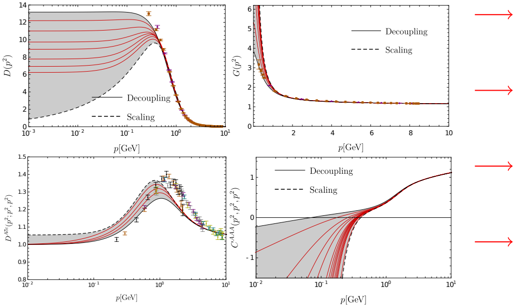
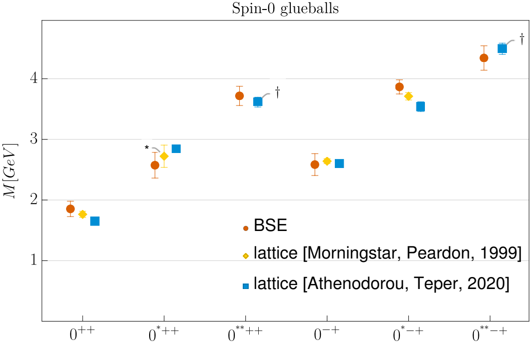


Unique physical spectrum:



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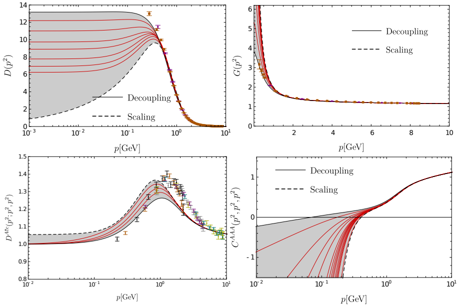

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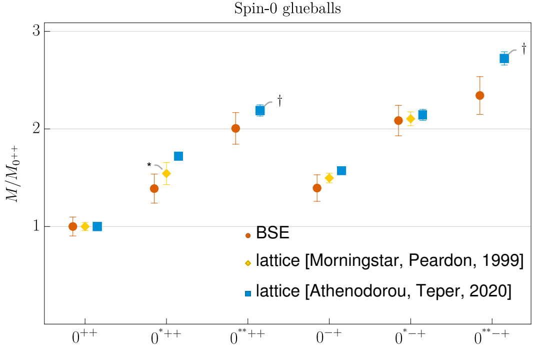


Spectrum independent! → Family of solutions yields the same physics.

All results for $r_0 = 1/418(5)$ MeV.

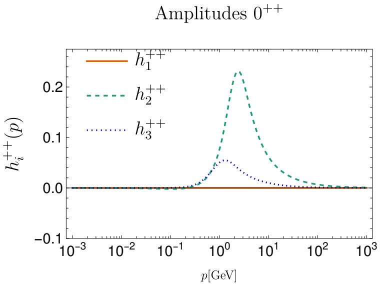
[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C80 (2020)]

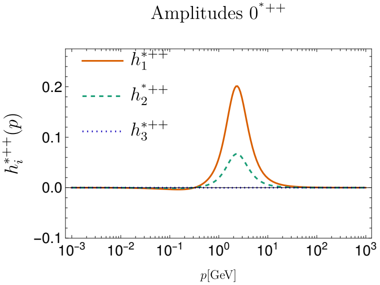
Amplitudes

Information about significance of single parts.

Ground state scalar glueball:

Excited scalar glueball:





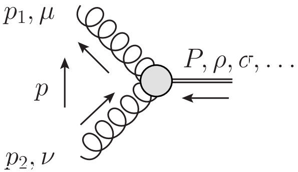
→ Amplitudes have different behavior for ground state and excited state. Useful guide for future developments.

→ Meson/glueball amplitudes: Information about mixing.

Glueball amplitudes for spin J

[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C81 (2021)]

$$\Gamma_{\mu\nu\rho\sigma\ldots}(p_1,p_2) = \sum \tau^i_{\mu\nu\rho\sigma\ldots}(p_1,p_2) h_i(p_1,p_2)$$



Increase in complexity:

- 2 gluon indices (transverse)
- J spin indices (symmetric, traceless, transverse to P)

Numbers of tensors:

J	P = +	P = −
0	2	1
1	4	3
>2	5	4

Low number of tensors, but high-dimensional tensors!

$\rightarrow$ Computational cost increases with J.

$J = 1$ glueballs

Landau-Yang theorem

Two-photon states cannot couple to $J^P = \mathbf{1}^{\pm}$ or $(2n+1)^-$

[Landau, Dokl.Akad.Nauk SSSR 60 (1948); Yang, Phys. Rev. 77 (1950)].

($\rightarrow$ Exclusion of $J = 1$ for Higgs because of $h \rightarrow \gamma\gamma$.)

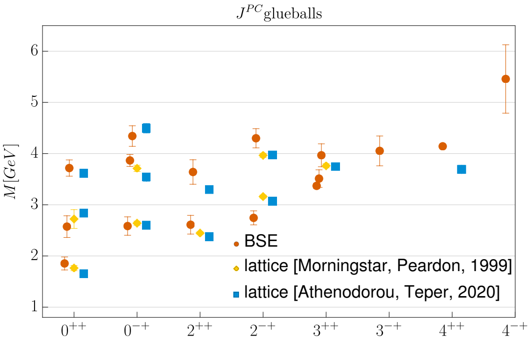
Applicable to glueballs?

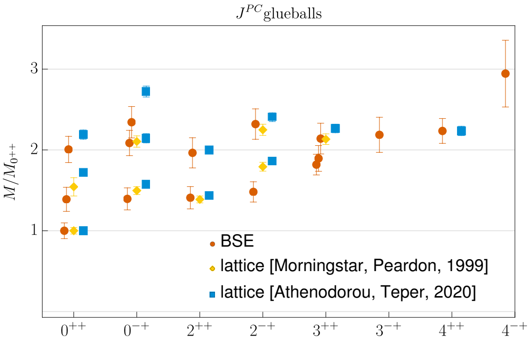
$\rightarrow$ Not in this framework, since gluons are not on-shell.

$\rightarrow$ Presence of $J = 1$ states is a dynamical question.

$J = 1$ not found here.

Glueball results



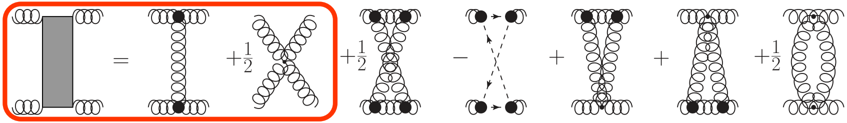


[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C81 (2021)]

- Agreement with lattice results
- New states: 0^{**++}, 0^{**-+}, 3^{-+}, 4^{-+}

Higher order diagrams

Testing an extension of the bound state equation: more diagrams in kernels



One-loop diagrams only:

[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C80 (2020); MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C81 (2021)]

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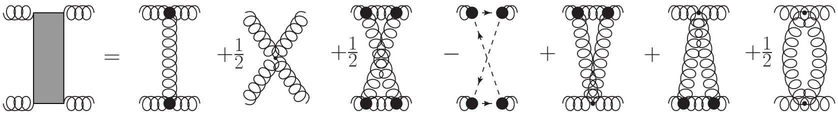

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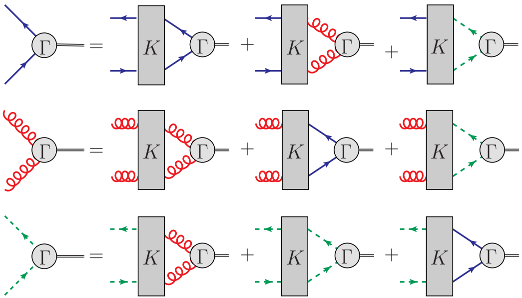
Two-loop diagrams: subleading effects

- 0^{-+}: none
  [MQH, Fischer, Sanchis-Alepuz, EPJ Web Conf. 258 (2022)]
- 0^{++}: $< 2\%$
  [MQH, Fischer, Sanchis-Alepuz, HADRON2021, arXiv:2201.05163]
- 2^{++}: none
  [MQH, Fischer, Sanchis-Alepuz, HADRON2023, arXiv:2312.12029]

Summary and conclusions

Functional bound state equations

- Tool for hadron physics: From qualitative insights to quantitative results



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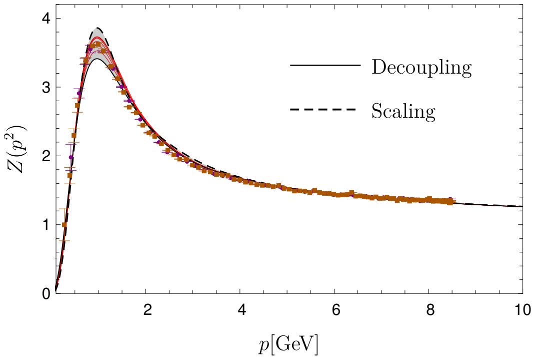

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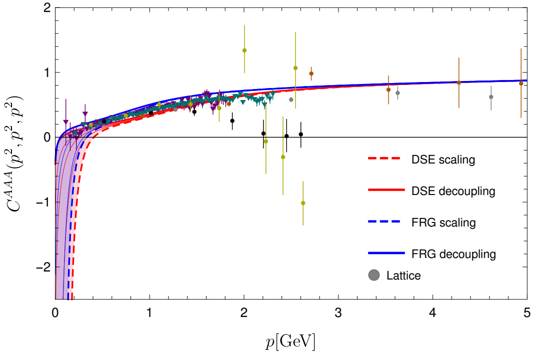

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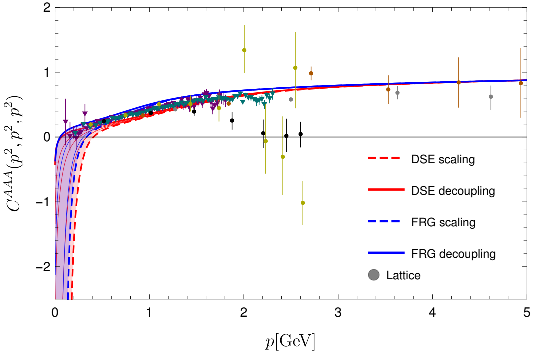

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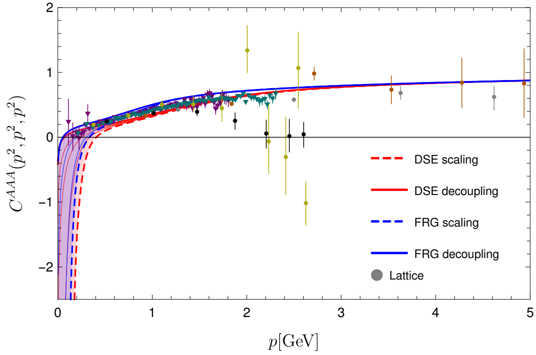

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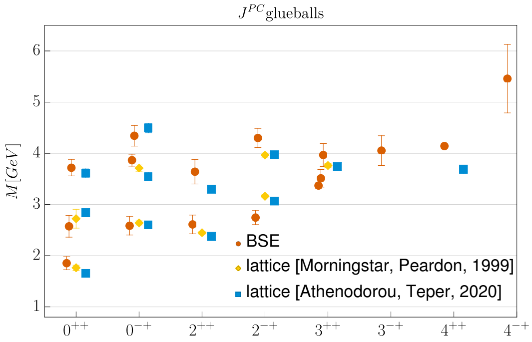
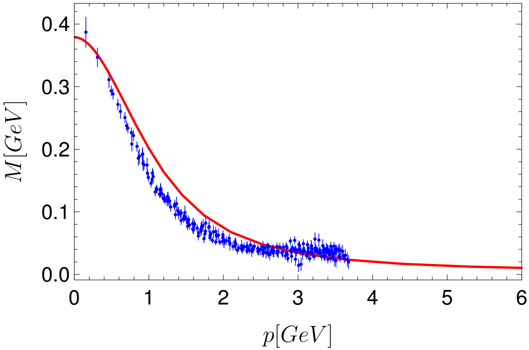
- Glueballs of pure gauge theory



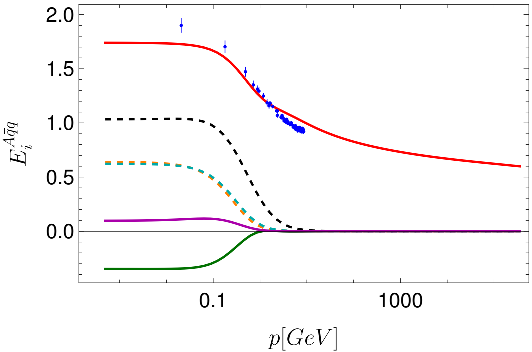
Outlook

- QCD:



Outlook

- QCD:
  - Quark sector



Outlook

- QCD:
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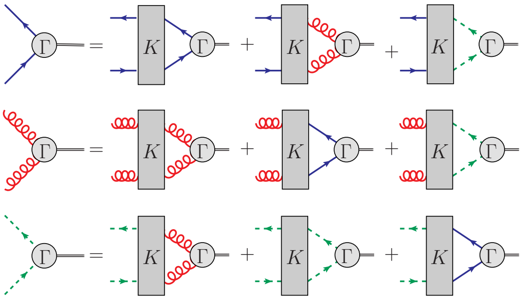

Outlook

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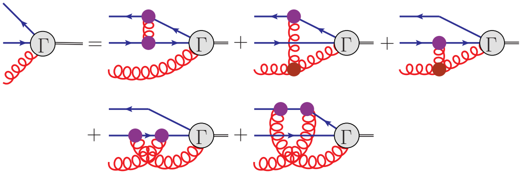
  - Quark sector
  - Mixing
- Three-body bound state equation:



Outlook

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  - Mixing
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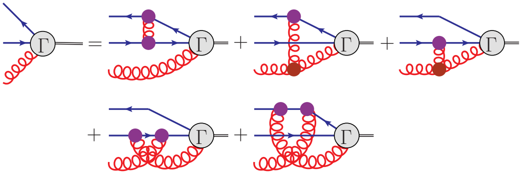
  - $C = -1$ glueballs



Outlook

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  - Mixing
- Three-body bound state equation:
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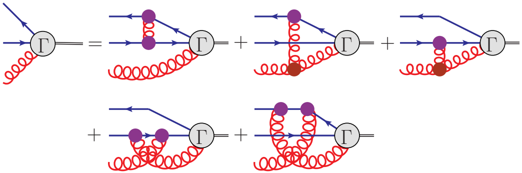
  - Hybrids



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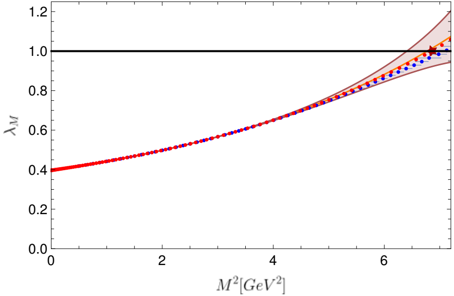
  - $C = -1$ glueballs
  - Hybrids
- Correlation functions in the complex plane:



Outlook

- QCD:
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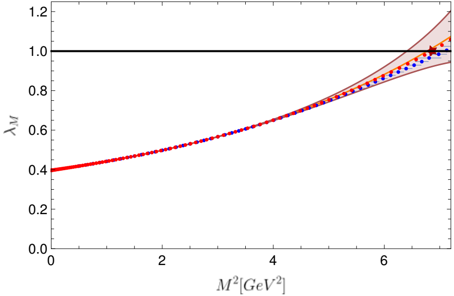

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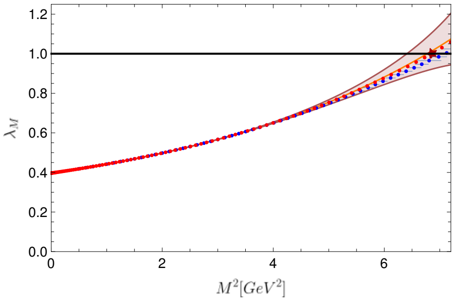

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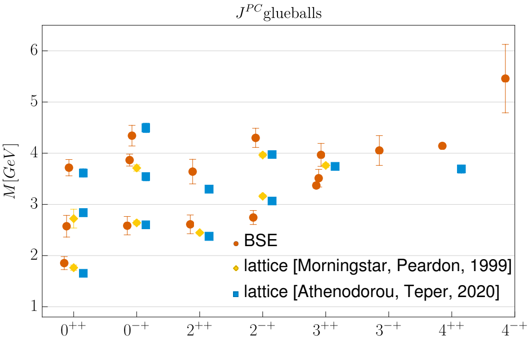


Thank you for your attention!

Charge parity

Transformation of gluon field under charge conjugation:

$$A^a_\mu \to -\eta(a) A^a_\mu$$

where

$$\eta(a) = \begin{cases} +1 & a = 1,3,4,6,8 \\ -1 & a = 2,5,7 \end{cases}$$

Color neutral operator with two gluon fields:

$$A^a_\mu A^a_\nu \to \eta(a)^2 A^a_\mu A^a_\nu = A^a_\mu A^a_\nu.$$

$\Rightarrow C = +1$

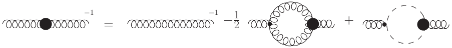
Negative charge parity, e.g.:

$$d^{abc} A^a_\mu A^b_\nu A^c_\rho \to - d^{abc} \eta(a)\eta(b)\eta(c) A^a_\mu A^b_\nu A^c_\rho = \\ - d^{abc} A^a_\mu A^b_\nu A^c_\rho.$$

Only nonvanishing elements of the symmetric structure constant d^{abc}: zero or two indices equal to 2, 5 or 7.

Landau gauge propagators in the complex plane

Simpler truncation:



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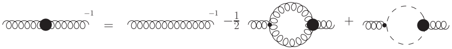

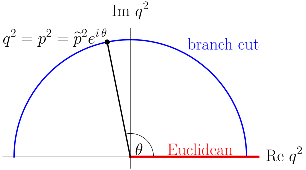

$\rightarrow$ Opening at $q^2 = p^2$.

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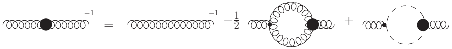

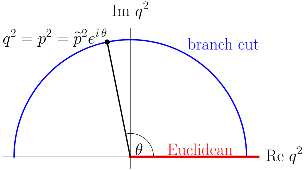

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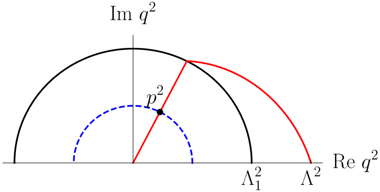
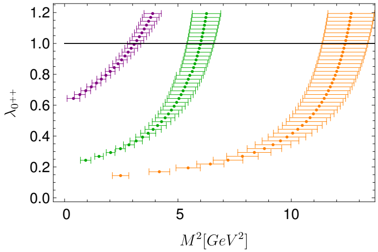


Appearance of branch cuts for complex momenta forbids integration directly to cutoff.

Extrapolation for glueball eigenvalue curves

0^{++}:



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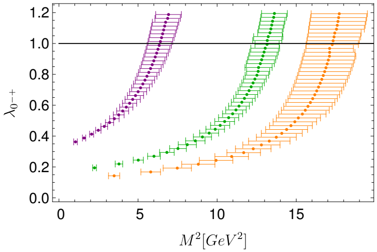

Several curves: ground state and excited states.

Bound state equations for hybrids

[Münster, Fischer, MQH]

- (Anti)quarks + gluonic excitation
- Meson $\rightarrow$ three-body equation
- Baryon $\rightarrow$ four-body equation

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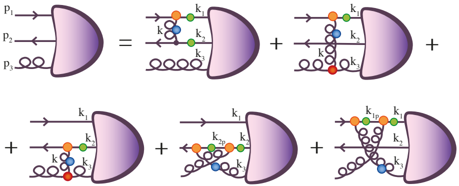

- $\pi_1(1600)$ (1^{-+}): 48 tensors
- Leading order of 3PI effective action: dressed quark-gluon and three-gluon interactions
- Preliminary results: diagrams with three-gluon vertices leading

Bound states in QCD

Hadron masses from correlation functions of color singlet operators (confinement!).

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$J^{PC} = 0^{-+}$ meson $\rightarrow O(x) = \overline{\psi}(x)\gamma_5\psi(x)$
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Lattice: Mass from exponential Euclidean time decay

$$\lim_{t\to\infty}\langle O(x)O(0)\rangle \sim e^{-t\,M}$$

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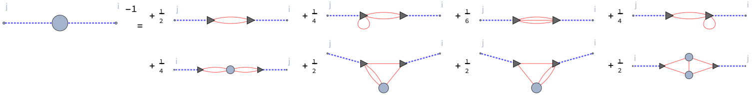
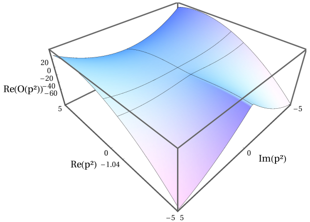
$$\boldsymbol{D(x-y)} = \langle O(x)O(y)\rangle$$

Functional approach:
Glueball:



+ 3-loop diagrams [MQH, Cyrol, Pawlowski, Comput.Phys.Commun. 248 (2020)]



Leading order: [Windisch, MQH, Alkofer, Phys.Rev.D87 (2013)]

Functional spectrum calculations: Top-down

Derivation of kernels and correlation functions from *n*PI effective actions [Fukuda, Prog.Theor.Phys. 78 (1987); Sanchis-Alepuz, Williams, J.Phys.Conf.Ser. 631 (2015)].

Loop expansion of *n*PI effective actions as reliable expansion in terms of nonperturbative quantities?

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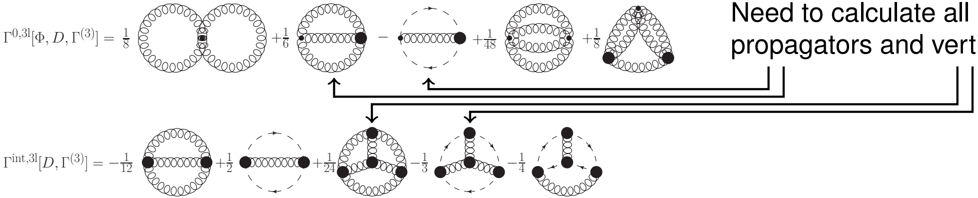
Example: 3-loop 3PI effective action [Berges, Phys. Rev. D 70 (2004); Carrington, Gao, Phys. Rev. D 83 (2011)]

$$\Gamma^{3l}[\Phi, D, \Gamma^{(3)}] = \Gamma^{0,3l}[\Phi, D, \Gamma^{(3)}] + \Gamma^{\text{int},3l}[\Phi, D, \Gamma^{(3)}]$$

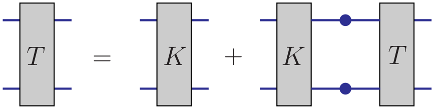


Need to calculate all propagators and vertices.

Functional bound state equation

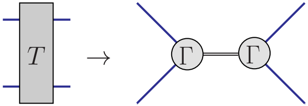
Dyson equation: nonperturbative resummation!

Compare: $f(x) = \frac{1}{1-x} = 1 + x + x^2 + \ldots = 1 + x\,f(x) = 1 + x + x^2\,f(x)$



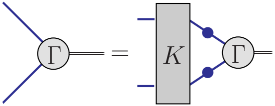
Scattering kernel K: interactions
Scattering matrix T
Bethe-Salpeter amplitude Γ

T contains bound states:



$$T \to \frac{\Gamma\,\overline{\Gamma}}{P^2 + M^2}$$

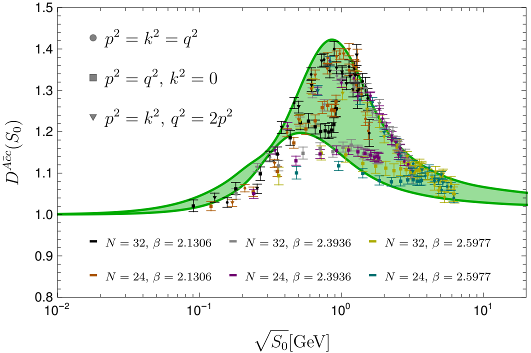
Plug into Dyson equation:
→ homogeneous Bethe-Salpeter equ.



[Review: Eichmann, Sanchis-Alepuz, Williams, Alkofer, Fischer, Prog.Part.Nucl.Phys. 91 (2016)]

Ghost-gluon vertex

Ghost-gluon vertex:



[Maas, SciPost Phys. 8 (2019);
MQH, Phys. Rev. D 101 (2020)]

- Nontrivial kinematic dependence of ghost-gluon vertex
- Qualitative agreement with lattice results, though some quantitative differences (position of peak!).