Dyson-Schwinger Equations in the Maximally Abelian Gauge

R. Alkofer Markus Q. Huber K. Schwenzer

Department of Physics, Karl-Franzens University Graz

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SIC!QFT

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Methods in Qantum ChromoDynamics

- Perturbation theory (expansion around small parameter; in QCD valid for high momenta)
- Monte Carlo simulations on a lattice (discretization of space-time; upper and lower bounds on momenta by lattice size and spacing)
- Effective theories and models
- Functional methods (Green functions):
  - Functional renormalization group
  - n-PI action
  - Stochastic quantization
  - Dyson-Schwinger equations (DSEs)

The Lagrangian of QCD

QED: $$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \mathcal{L}_{matter}$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Invariant under gauge transformation $\Omega(x)$ (rotation in color space):

$$A_\mu(x) \to A'_\mu(x) = \Omega(x) A_\mu(x) \Omega^\dagger(x)$$

$\Rightarrow$ Equivalent field configurations exist.

The Lagrangian of QCD

QCD: $$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \mathcal{L}_{matter}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i \left[A_\mu, A_\nu \right]$$

Yang-Mills theory: No quarks, only gauge fields.

Invariant under gauge transformation $\Omega(x)$ (rotation in color space):

$$A_\mu(x) \to A'_\mu(x) = \Omega(x) A_\mu(x) \Omega^\dagger(x) + i \left(\partial_\mu \Omega(x) \right) \Omega^\dagger(x)$$

$\Rightarrow$ Equivalent field configurations exist.

Propagator

Why fix the gauge?

2-point functions cannot be inverted, e.g. photon propagator:

Take inverse of quadratic part of action.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} = \frac{1}{2}\left((\partial_\mu A_\nu)^2 - (\partial_\mu A_\nu)(\partial_\nu A_\mu)\right)$$
$$\xrightarrow{part.int.} \frac{1}{2} A_\mu (-\Box \boldsymbol{\delta_{\mu\nu}} + \boldsymbol{\partial_\mu \partial_\nu}) A_\nu$$

This operator has no inverse!

Gauge Fixing: Linear Covariant Gauges

Lorenz gauge condition: $\partial A = 0$

$\rightarrow$ Add $+\frac{1}{2\xi}(\partial_\mu A_\mu)^2$ to Lagrangian.

Inverse $\rightarrow \Delta_{\mu\nu} = (-\delta_{\mu\nu} + (1-\xi)\Box^{-1}\partial_\mu\partial_\nu)\frac{1}{\Box}$

Fourier transformation:

$$\Delta_{\mu\nu}(p) = \left(\delta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2}$$

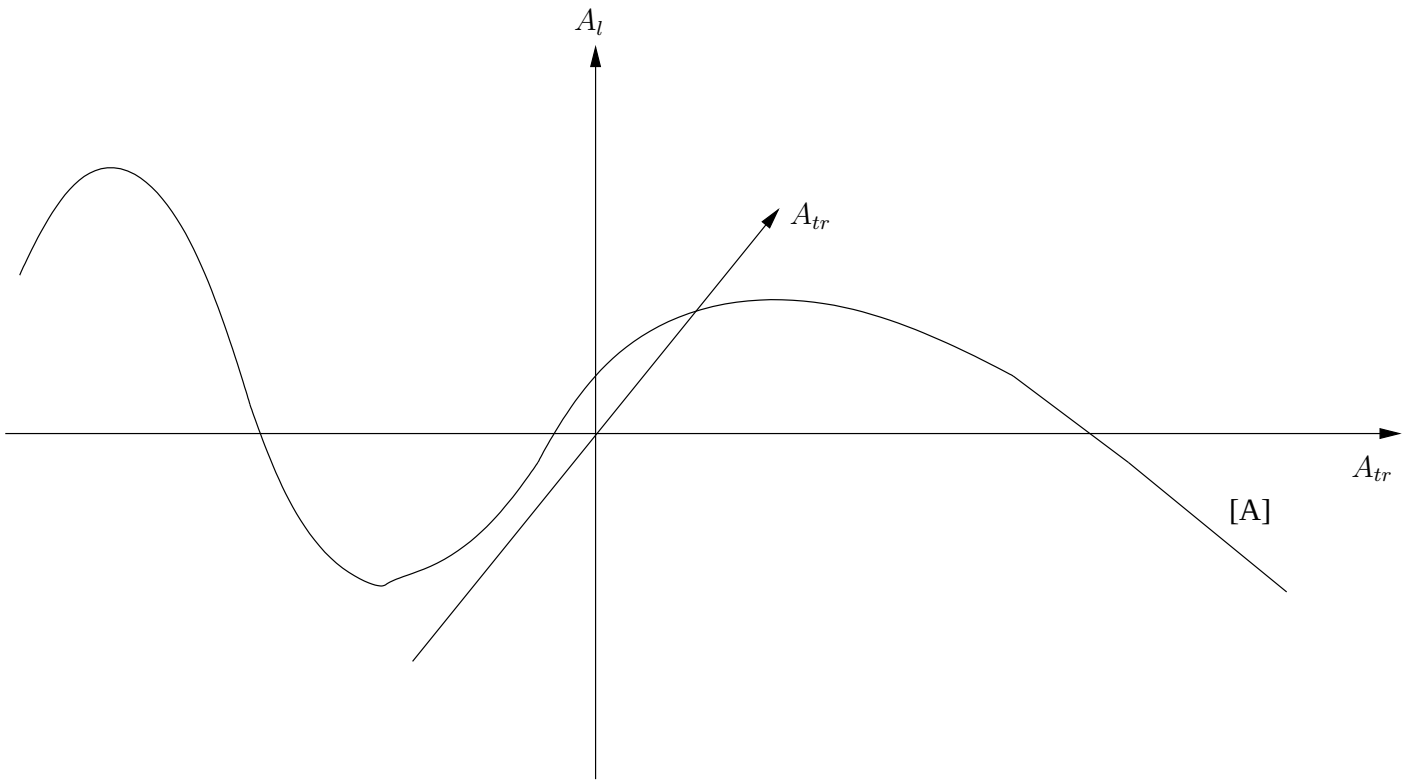
- $\xi = 0$: Landau gauge
- $\xi = 1$: Feynman gauge, important in perturbation theory

$\Rightarrow$ Green functions are gauge dependent.

Gauge Fixing in Yang-Mills Theory

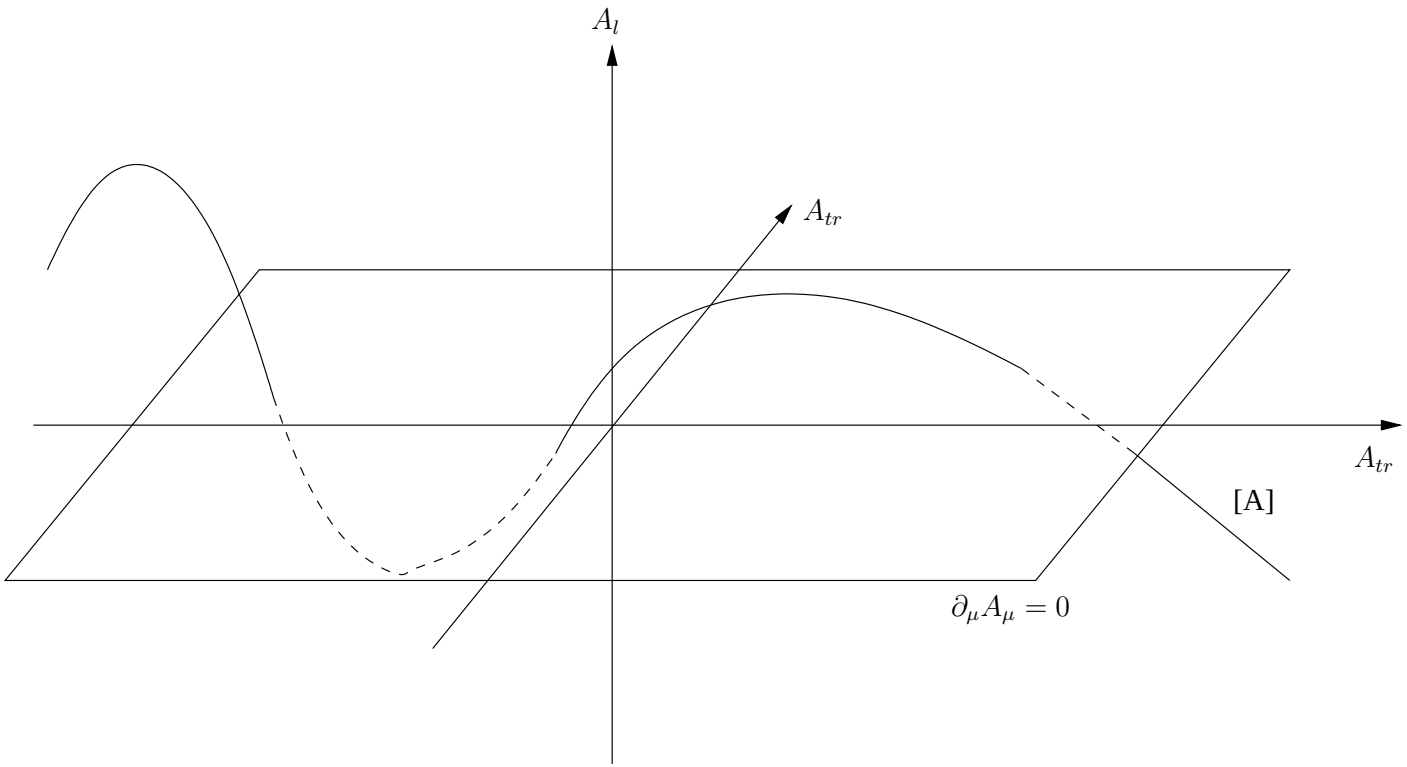
- Gauge transf. connect: equivalent configurations $\longrightarrow$ gauge orbit $[A]$.
- Integration in path integral over all A_μ is overcomplete.
- Idea by Faddeev and Popov: Reduce integration to single representative of each gauge orbit.
- Gauge fixing term: $\frac{1}{2\xi}(\partial_\mu A_\mu)^2 + \bar{c}\, M\, c$
- ghost fields: wrong statistic (even spin, but behave like fermions)
- $\Rightarrow$ 2 fields: gluons and ghosts

Gauge Orbits and Gribov Copies



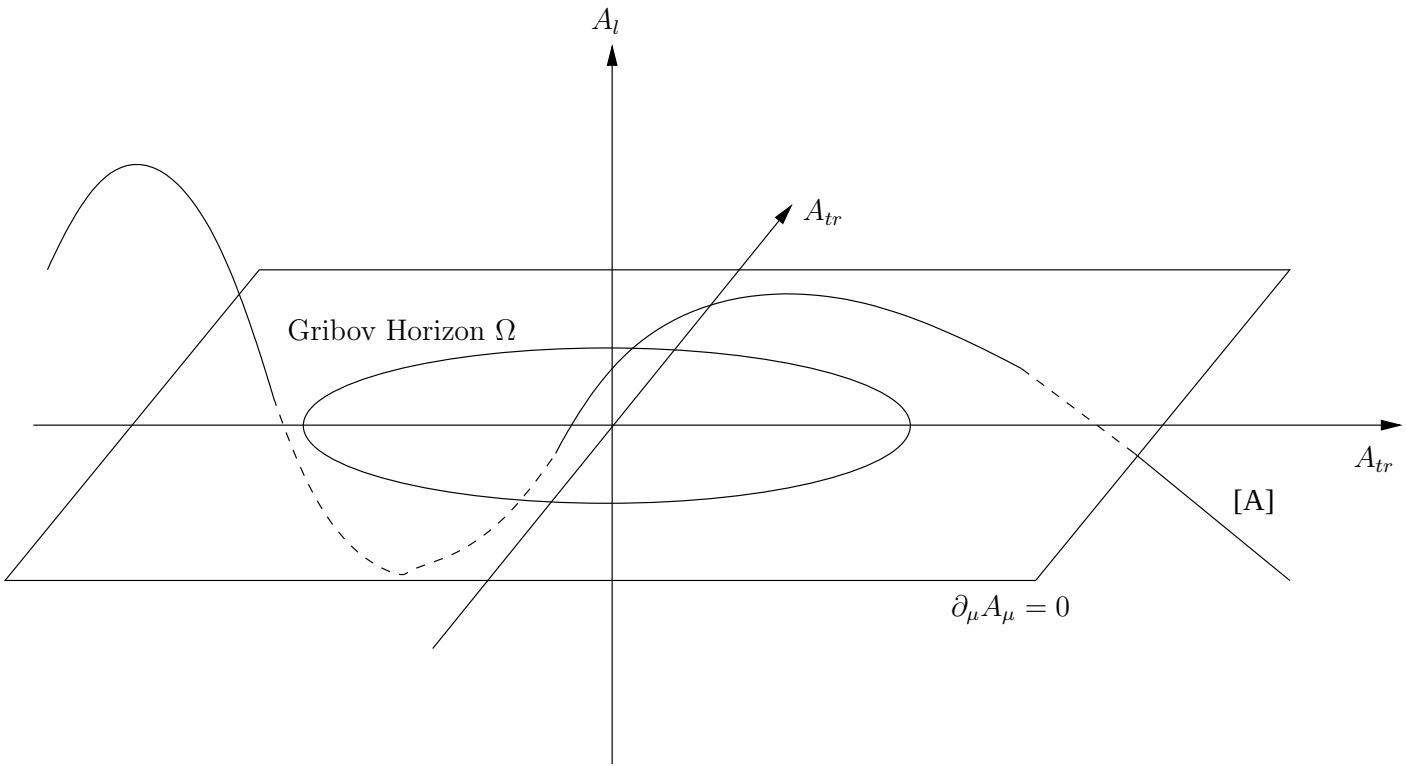
Fix the gauge to Landau gauge $\partial A = 0$.

Gauge Orbits and Gribov Copies



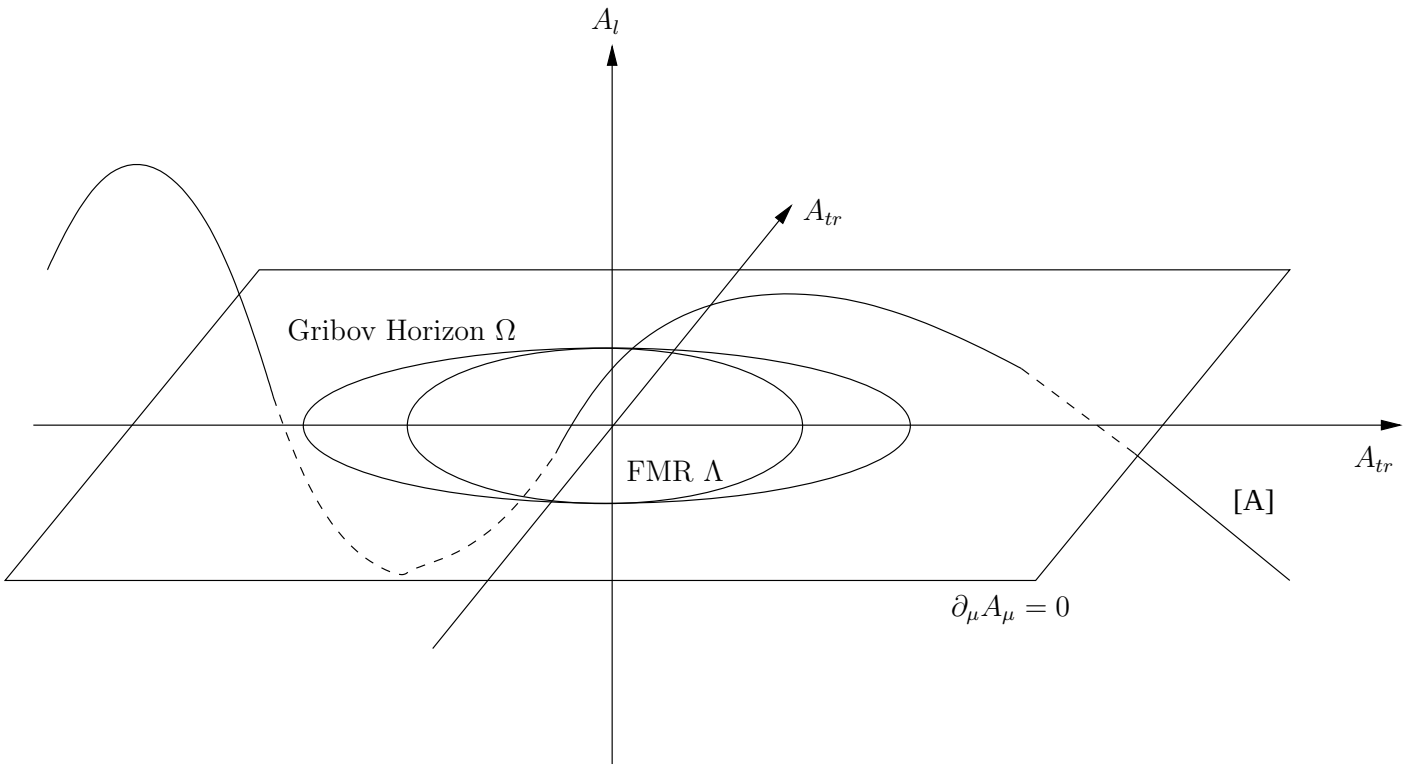
Gauge orbit should intersect hyperplane $\partial A = 0$ only once.

Gauge Orbits and Gribov Copies



Minimize some functional to get only one gauge configuration per orbit $\longrightarrow$ Gribov region.

Gauge Orbits and Gribov Copies



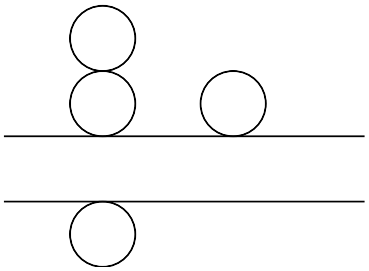
Fundamental modular region (FMR): absolute minimum of some fuctional.

Green Functions

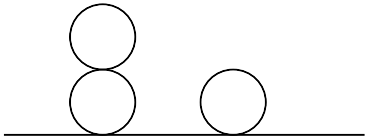
Green functions describe propagation and interaction of particles
$\longrightarrow$ propagators and vertices.

Derived from generating functionals (full, connected, 1PI) by differentiation.

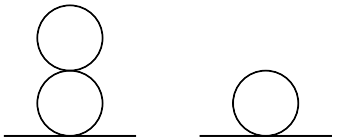
$$Z[J] = \sum_m \frac{1}{m!} G_m^{(f)} J^m$$



$$W[J] = \sum_m \frac{1}{m!} G_m^{(c)} J^m$$



$$\Gamma[\phi] = \sum_m \frac{1}{m!} G_m \phi^m$$



$$\frac{\delta^n \Gamma}{\delta\phi_i \ldots \delta\phi_n}$$

n-fold derivative
$\longrightarrow$ n-point functions:

- n=2: inverse of propagators
- n>2: vertices

Dyson-Schwinger Equations (DSEs) for Investigating QCD

DSE describe non-perturbatively how particles propagate and interact.

Facts about DSEs

- F. J. Dyson (1949) and J. S. Schwinger (1951)
- Equations of motion of Green functions
- Infinitely large tower of equations (DSE for n-point function contains n+1- and n+2-point functions)

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- Continuum
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Cons?:

- Truncations (not for all tasks)
- Gauge-dependent
  $\longrightarrow$ Exploit advantages of different gauges

Power Laws for Dressing Functions

In DSEs there are bare, like in perturbation theory, and dressed quantities $\rightarrow$ dressing functions.

- Dressing functions in the infrared (IR) can be described by power laws, e.g. for ghost propagator

$$D_G = -\frac{G(p^2)}{p^2}, \qquad G(p^2) \overset{p^2 \to 0}{=} B \cdot (p^2)^{\delta_{gh}}$$

- IR exponent: $\delta_{gh} > 1 \rightarrow$ propagator vanishes, $\delta_{gh} < 0 \rightarrow$ propagator IR enhanced.
- If external momenta small, integral dominated by small momenta.
- Upon integration loop momenta are transformed into external momenta.

Power Counting

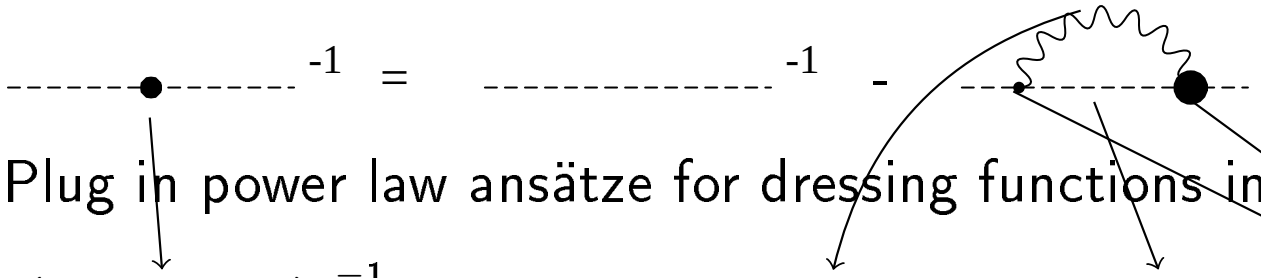
- The ghost propagator DSE:

- Plug in power law ansätze for dressing functions in the IR

$$\left(\frac{B \cdot (p^2)^\beta}{p^2} \right)^{-1} \sim \int \frac{d^d q}{(2\pi)^d} \, P_{\mu\nu} \frac{A \cdot (q^2)^\alpha}{q^2} \, \frac{B \cdot ((p-q)^2)^\beta}{(p-q)^2} \, (p-q)_\mu q_\nu$$

Power Counting

- The ghost propagator DSE:



- Plug in power law ansätze for dressing functions in the IR

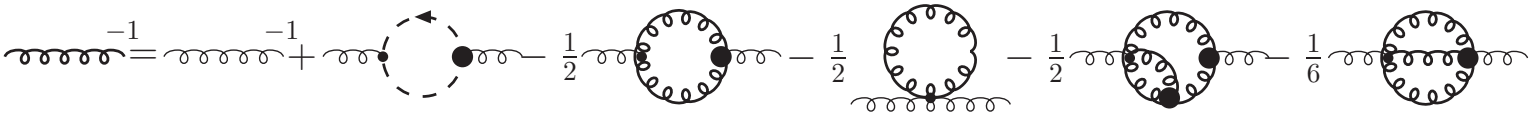
$$\left(\frac{B\cdot(p^2)^\beta}{p^2}\right)^{-1} \sim \int \frac{d^dq}{(2\pi)^d}\, P_{\mu\nu}\, \frac{A\cdot(q^2)^\alpha}{q^2}\, \frac{B\cdot((p-q)^2)^\beta}{(p-q)^2}\, (p-q)_\mu q_\nu$$

- Only one momentum scale
  $\longrightarrow$ simple power counting is possible:
- $1-\beta = \frac{d}{2} + \alpha - 1 + \beta - 1 + \frac{1}{2} + \frac{1}{2} \Longrightarrow -2\beta = \alpha + \frac{d}{2} - 2$

Constraints from DSEs

Still, at the end there is a big system of inequalities that has to be solved, e.g. gluon propagator in Landau gauge:

$$-\delta_g = \min(\underbrace{0}_{\text{bare prop.}}, \underbrace{2\delta_g + \delta 3g}_{\text{gh loop}}, \underbrace{2\delta_{gh} + \delta_{gg}}_{\text{gl loop}}, \underbrace{\delta_g}_{\text{tadpole}}, \underbrace{4\delta_g + 2\delta_{3g}}_{\text{squint}}, \underbrace{3\delta_g + \delta_{4g}}_{\text{sunset}})$$



min-function $\Rightarrow$ set of inequalities,
e.g. from sunset $4\delta_g + \delta_{4g} \geq 0$.
From four-gluon DSE:
$\delta_{4g} \leq 0 \Rightarrow \delta_g \geq 0$.

IR Exponent for Arbitrary Diagram

System difficult to solve. Look for alternatives or shortcuts.

Arbitrary Diagram v

Numbers of vertices and propagators related $\Rightarrow$ possible to get a formula for the IR exponent.

Function of:

- propagator IR exponents δ_{X_i}
- number of external legs m^{X_i}
- number of vertices.

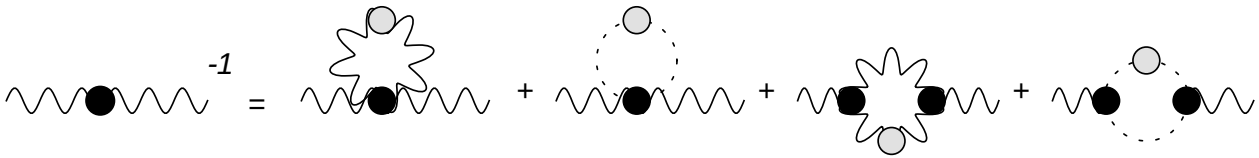
$$\delta_v = -\frac{1}{2}\sum_i m^{X_i}\delta_{X_i} + \\ + \sum_i (\#\text{ of dressed vertices})_i C_1^i + \sum_i (\#\text{ of bare vertices})_i C_2^i$$

Functional Renormalization Group

Functional equations similar to DSEs, but with decisive differences:

- only 1-loop diagrams
- all quantities dressed

Renormalization group equations (RGEs) are "differential DSEs".



Restrictions

Combining DSEs and RGEs (idea by Fischer, Pawlowski, 2006)


+

=


we get bounds for coefficients C_1^i and C_2^i:

$$C_1^i \geq 0, \qquad\qquad C_2^i \geq 0$$

$\Rightarrow$ Maximally IR divergent solution: $\delta_{v,max} = -\frac{1}{2}\sum_i m^{X_i}\delta_{X_i}$.

Only depends on number and type of external fields.

Existence of Scaling Solutions

Abundance of inequalities → reduce to relevant ones. Only small number left, e. g. Landau gauge:

$$\delta_{gh} + \frac{1}{2}\delta_g \geq 0, \quad \delta_g \geq 0$$

Looking for solutions of the propagator DSEs → at least one of these equations has to be saturated:

- $\delta_g = 0$: corresponds to trivial solution (perturbation theory)
- $2\delta_{gh} = -\delta_g$: scaling solution for Landau gauge [von Smekal, Hauck, Alkofer, 1998]

Condition of saturation of one inequality: Restricts naive existence of scaling solutions in other gauges, e. g. covariant gauges, ghost anti-ghost symmetric gauges.

The Maximally Abelian Gauge (MAG) in $SU(2)$

Split the gluon field $A_\mu = A^r_\mu T^r$
into diagonal and off-diagonal fields:

$$A_\mu = B^a T^a + A^i T^i$$

T^a: off-diagonal matrices
T^i: diagonal matrices
$\Rightarrow$ Gluonic vertices split: ABB, $AABB$, $BBBB$
Number of diagonal fields per vertex restricted.

Non-linear gauge fixing

Minimize off-diagonal components along gauge orbit:

$$D^{ab}_\mu B^b = (\delta_{ab}\partial_\mu - g\, f^{abi} A^i_\mu) B^b = 0$$

$\Rightarrow$ ghosts and interactions: Acc, $AAcc$, $BBcc$

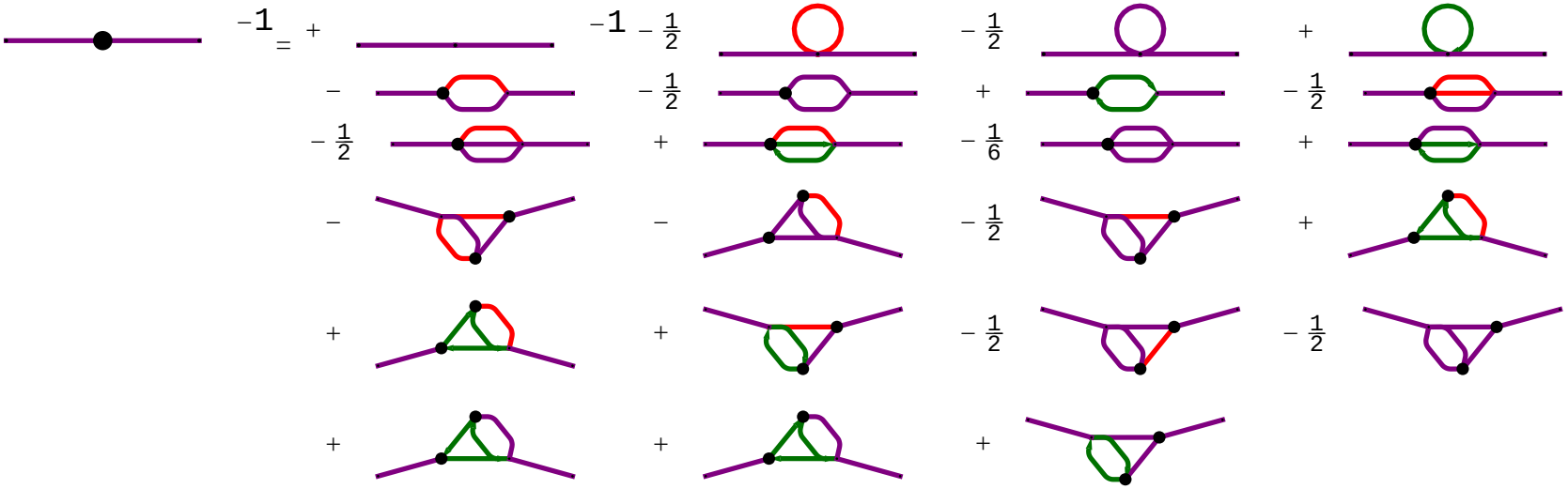
builing blocks for DSEs: ABB, Acc, $AABB$, $AAcc$, $BBcc$, $BBBB$, $cccc$

DSEs of the MAG

diagonal gluon: off-diagonal gluon:

off-diagonal ghost: 

off-diagonal gluon propagator DSE:



Constraints on Infrared Exponents in the MAG for $SU(2)$

Relevant inequalities:

$$\delta_{BBBB} + 2\delta_B \geq 0, \qquad \delta_{cccc} + 2\delta_c \geq 0,$$
$$\delta_{AABB} + \delta_A + \delta_B \geq 0, \qquad \delta_{AAcc} + \delta_A + \delta_c \geq 0,$$
$$\delta_B \geq 0, \qquad \delta_c \geq 0,$$
$$\delta_A + \delta_B \geq 0, \qquad \delta_A + \delta_c \geq 0,$$

$$\delta_{BBcc} + \delta_B + \delta_c \geq 0, \qquad \delta_B + \delta_c \geq 0,$$
$$\delta_{ABB} + \frac{1}{2}\delta_A + \delta_B \geq 0, \qquad \frac{1}{2}\delta_A + \delta_B \geq 0,$$
$$\delta_{Acc} + \frac{1}{2}\delta_A + \delta_c \geq 0, \qquad \frac{1}{2}\delta_A + \delta_c \geq 0$$

Constraints on Infrared Exponents in the MAG for $SU(2)$

Relevant inequalities:

$$\delta_{BBBB} + 2\delta_B \geq 0, \qquad \delta_{cccc} + 2\delta_c \geq 0,$$

$$\delta_{AABB} + \delta_A + \delta_B \geq 0, \qquad \delta_{AAcc} + \delta_A + \delta_c \geq 0,$$

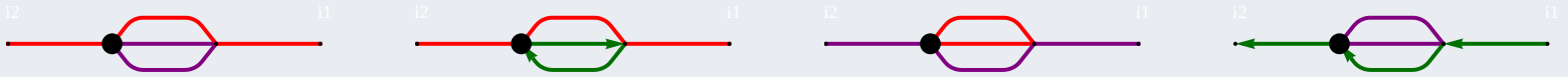
$$\boldsymbol{\delta_B \geq 0}, \qquad \boldsymbol{\delta_c \geq 0},$$

$$\delta_A + \delta_B \geq 0, \qquad \delta_A + \delta_c \geq 0,$$

Results for the MAG in $SU(2)$

Leading diagrams in propagator DSEs

All sunsets with 2 diagonal & 2 off-diagonal fields



possibly some squints

- IR enhanced diagonal gluon:

$$\delta_B = \delta_c =: \kappa \geq 0, \qquad \delta_A = -\kappa$$

- Supports Abelian dominance hypothesis (relevant degrees of freedom in diagonal part of gluon field).
- Vertices with 2 diagonal & 2 off-diagonal fields do NOT scale, i.e. $\delta_{AABB} = \delta_{AAcc} = 0$.

The MAG in $SU(3)$

In general there are more interactions than included above.
$\rightarrow$ Different solution for "physical system", i. e. $SU(3)$?

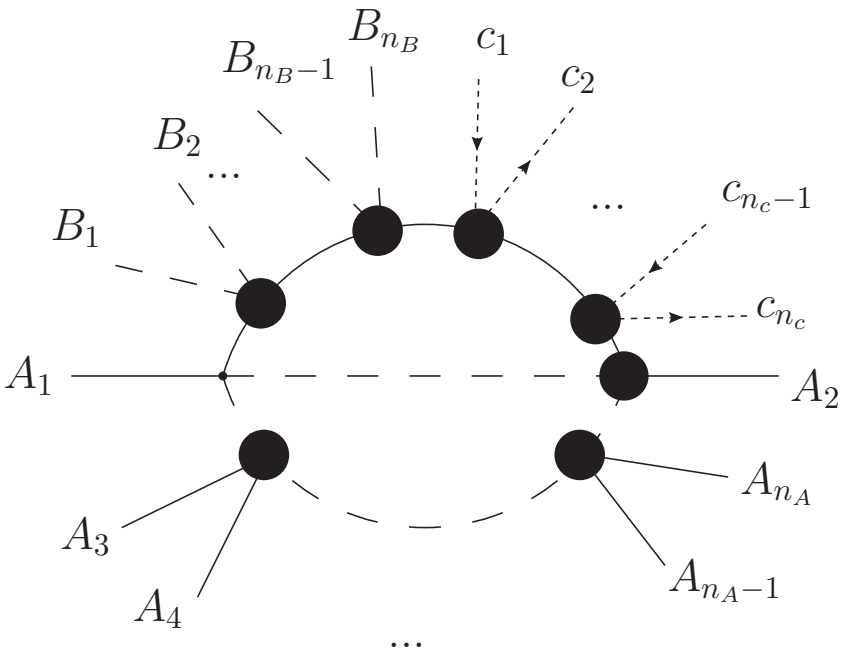
4 additional vertices: BBB, Bcc, $ABBB$, $ABcc$
Constraints:

$$\frac{3}{2}\delta_B \geq 0, \qquad \frac{1}{2}\delta_B + \delta_c \geq 0,$$
$$\frac{1}{2}\delta_A + \frac{3}{2}\delta_B \geq 0, \qquad \frac{1}{2}\delta_A + \frac{1}{2}\delta_B + \delta_c \geq 0$$

Already contained in "old" system $\rightarrow$ nothing new, solution still valid and unique.

Higher n-Point Functions

Sunsets definitely leading: $\longrightarrow$ successively add pairs of fields
$\longrightarrow$ n-point functions with n even



n odd: at least one vertex with an odd number of legs,
3cannot be determined uniquely.

Summary: Functional Approaches & Scaling Solutions

Functional approaches

- Gauge fixing necessary for functional approaches.
- Infinitely large tower of equations.
- Solution in the IR possible without truncations.

Scaling solutions

- Power laws for dressing functions with infrared exponents in IR regime.
- Maximally IR divergent solution depends on number of legs and propagator IR exponents.
- At least one vertex does not scale.
- Self-interacting fields (e.g. gluon in Landau gauge) have non-negative IR exponents $\Rightarrow$ no IR enhancement.

Summary: Results for the MAG

- $\delta_B = \delta_c =: \kappa \geq 0, \qquad \delta_A = -\kappa$
- Diagonal gluon ($\boldsymbol{A}$) IR enhanced
  $\Rightarrow$ supports Abelian dominance hypothesis.
- Off-diagonal fields ($\boldsymbol{B}$, $\boldsymbol{c}$) are IR suppressed (compared to tl.).
- Four-point functions with 2 diagonal and 2 off-diagonal fields do not scale $\Rightarrow$ **2-loop diagrams IR leading**.
- Unique solution for vertices with an even number of legs.
- For vertices with an odd number of legs several solutions.

- No qualitative difference between $SU(2)$ and $SU(3)$.