Infrared Analysis of Yang-Mills Green Functions

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SIC!QFT

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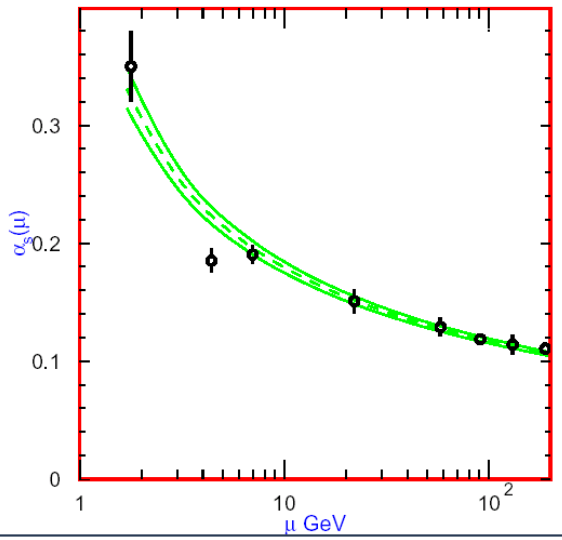
Nature and Qantum ChromoDynamics

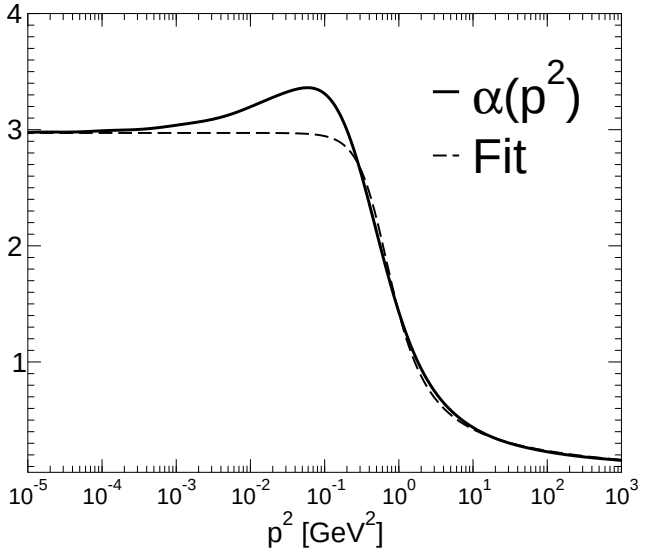
- Nature offers an abundance of particles called hadrons, which interact via the strong force:
  π, K, η, proton, neutron, Λ, Σ, ...
- Are these particles fundamental?
- Experiments (electron-proton scattering) show that hadrons seem to be built of point-like particles called quarks.
- Force holding quarks together in a bound state: strong interaction, mediated by gluons.
- Confinement: No free quarks and gluons are observed.
- Theory that describes the strong interaction:
  Quantum Chromodynamics (QCD)

Methods in Qantum ChromoDynamics

- Perturbation theory (expansion around small parameter; in QCD valid for high momenta $\longrightarrow$ asymptotic freedom, Nobel price 2004)
- Monte Carlo simulations on a lattice (discretization of space-time; upper and lower bounds on momenta by spacing and lattice size)
- Effective theories and models
- Functional methods (description in terms of Green functions):
  - Functional renormalization group
  - n-PI action
  - Stochastic quantization
  - Dyson-Schwinger equations (DSEs)



Rev. of particle physics, PLB 667 (2008)



Fischer, Alkofer, PLB 536 (2002)

Dyson-Schwinger equations (DSEs) for investigating QCD

DSE describe non-perturbatively how particles propagate and interact.

Facts about DSEs

- F. J. Dyson (1949) and J. S. Schwinger (1951)
- Equations of motion of Green functions
- Infinitely large tower of equations (DSE for n-point function contains n+1- and n+2-point functions)

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- Exact equations
  $\longrightarrow$ non-perturbative regime accessible
- Continuum
  $\longrightarrow$ complement lattice method

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Pros:

- Exact equations
  $\longrightarrow$ non-perturbative regime accessible
- Continuum
  $\longrightarrow$ complement lattice method

Cons?:

- Truncations (not for all tasks)
- Gauge-dependent
  $\longrightarrow$ Exploit advantages of different gauges

What does a Dyson-Schwinger equation look like?

Gluon propagator Dyson-Schwinger equation:

What we want to calculate: gluon propagator $D^{ab}_{\mu\nu}$.
What we need: ghost propagator G_{ab}, vertices.

What does a Dyson-Schwinger equation look like?

Gluon propagator Dyson-Schwinger equation:

$$
\begin{aligned}
D^{-1\,ab}_{\mu\nu}(p) &= Z_3\, D^{-1\,ab}_{(0)\,\mu\nu}(p) \\
&- Z_1 \frac{1}{2} \int d^4q\, d^4k\, \Gamma^{(0)\,acd}_{\mu\alpha\beta}(p,-q,-k)\, D^{de}_{\beta\gamma}(k)\, D^{cf}_{\alpha\delta}(q)\, \Gamma^{bef}_{\nu\gamma\delta}(-p,k,q) \\
&+ Z_4 \frac{1}{2} \int d^4q_1\, d^4q_2\, \Gamma^{(0)\,abcd}_{\mu\nu\alpha\beta}(p,-p,-q_1,q_2)\, D^{cd}_{\alpha\beta}(q_1) \\
&- Z_4 \frac{1}{6} \int d^4k_1\, d^4k_2\, d^4k_3\, \Gamma^{(0)\,acde}_{\mu\alpha\beta\gamma}(p,-k_1,-k_2,-k_3) \\
&\quad \times D^{cm}_{\alpha\lambda}(k_1)\, D^{dl}_{\beta\sigma}(k_2)\, D^{ek}_{\gamma\rho}(k_3)\, \Gamma^{bklm}_{\nu\rho\sigma\lambda}(p,k_3,k_2,k_1) \\
&+ Z_4 \frac{1}{2} \int d^4k_1\, d^4k_2\, d^4k_3\, d^4k_4 \\
&\quad \times \Gamma^{(0)\,acde}_{\mu\alpha\beta\gamma}(p,-k_1,-k_2,-k_3)\, D^{ck}_{\alpha\rho}(k_1)\, D^{dm}_{\beta\lambda}(k_2)\, D^{ep}_{\gamma\delta}(k_3) \\
&\quad \times \Gamma^{klm}_{\rho\sigma\lambda}(k_1,-k_4,k_2)\, D^{lq}_{\sigma\kappa}(k_4)\, \Gamma^{bpq}_{\nu\delta\kappa}(-p,k_3,k_4) \\
&- \widetilde{Z}_1 \int d^4q\, d^4k\, \Gamma^{(0)\,acd}_{\mu}(p,q,k)\, G^{de}(-q)\, G^{fc}(k)\, \Gamma^{bef}_{\nu}(-p,k,q)
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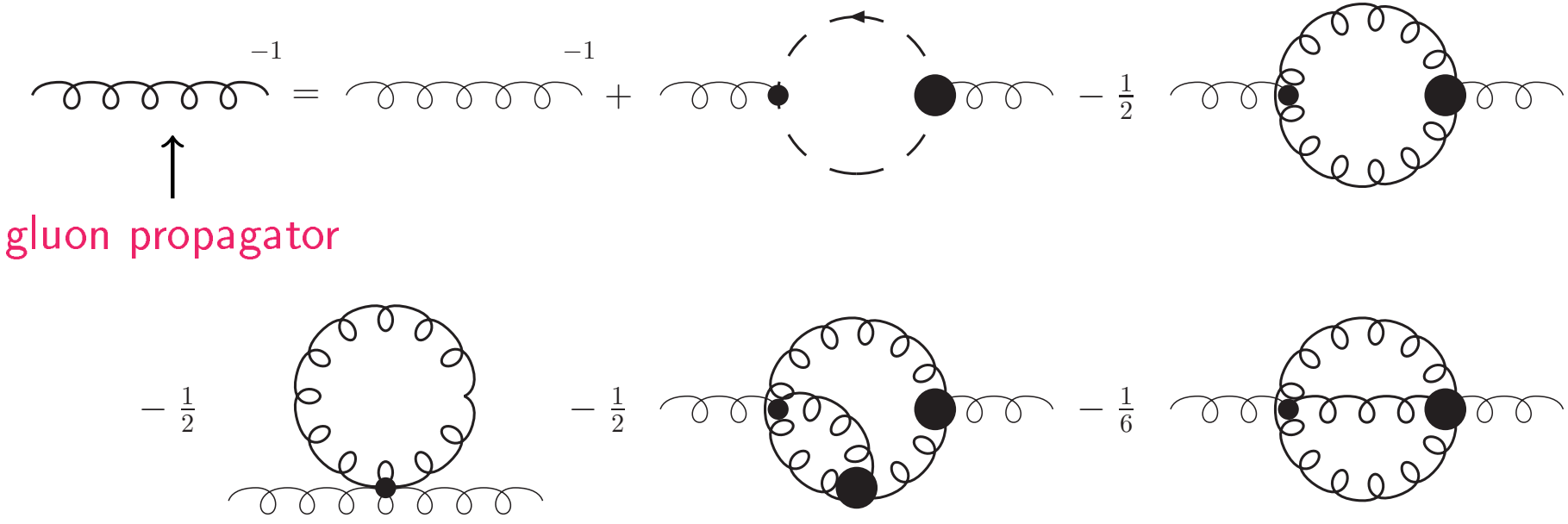

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- there are bare and dressed interaction vertices

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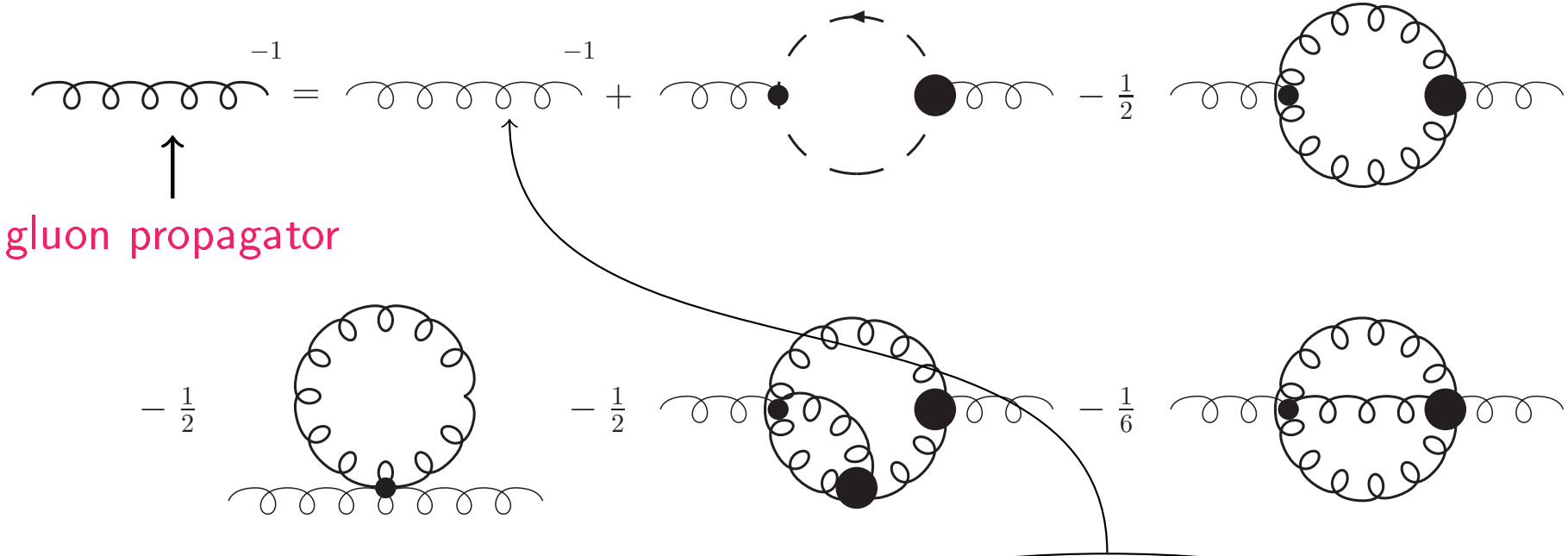

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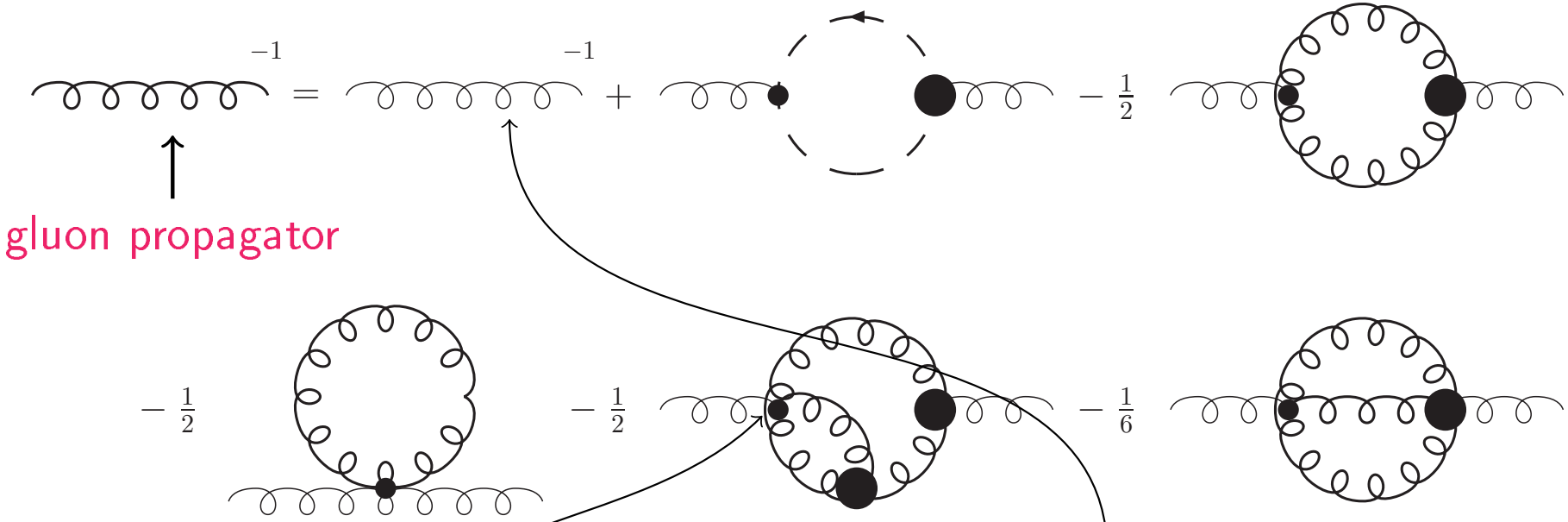

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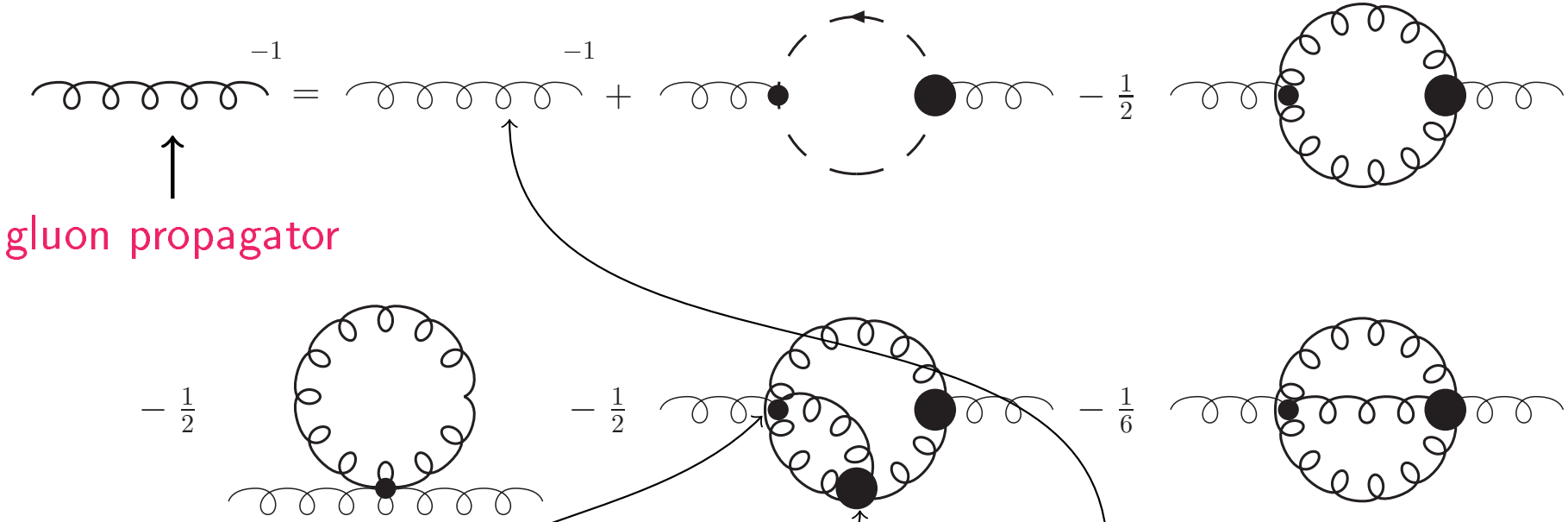

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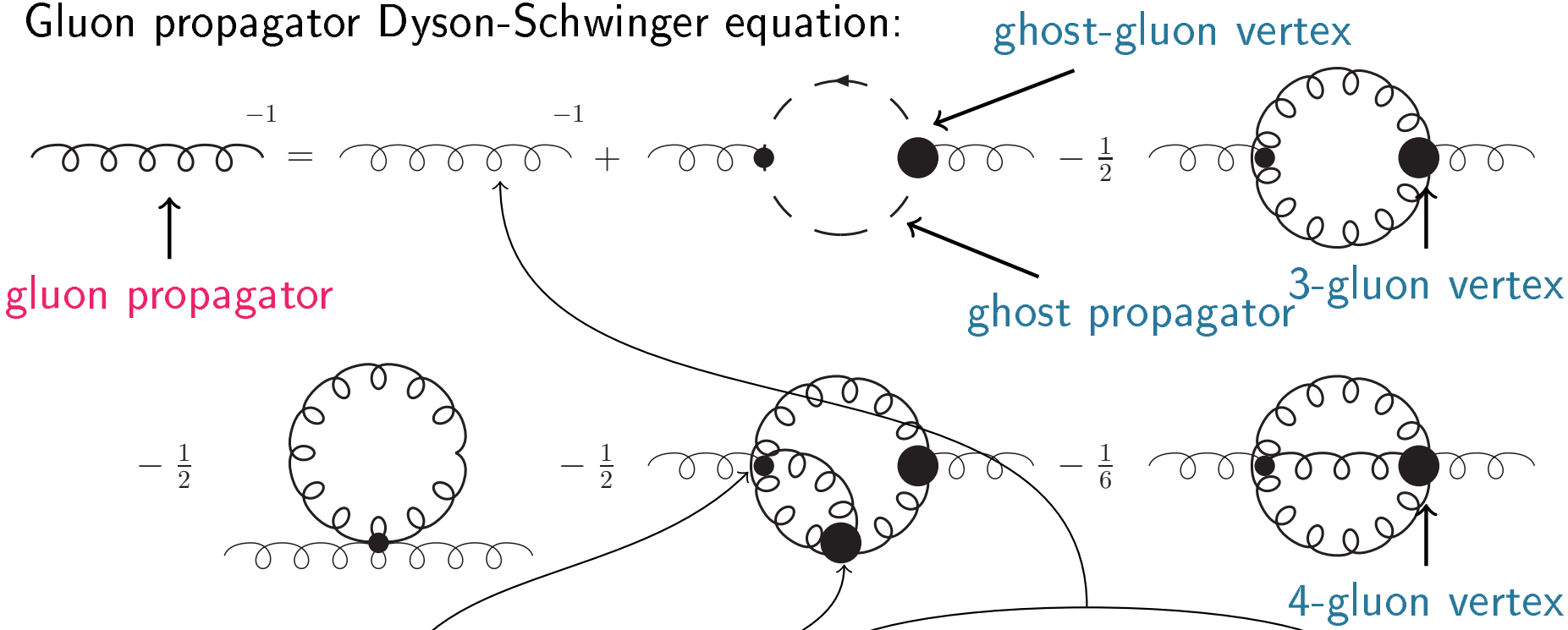

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The Lagrangian of QCD

Lagrangian of quarks (similar to QED):

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Field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\,g\,\boxed{[A_\mu, A_\nu]}$

$$\mathcal{L}_{YM} = \frac{1}{2} tr(F_{\mu\nu}F_{\mu\nu})$$

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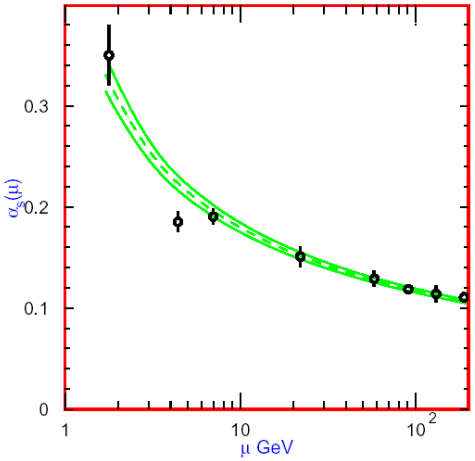
BUT: non-Abelian gauge group $\rightarrow$ self-interaction of the gauge field

non-Abelian gauge groups differ from Abelian ones qualitatively, e. g. can have asymptotic freedom and strong interaction in low momentum regime



PLB 667 (2008)

The path integral for Yang-Mills theory

Yang-Mills theory: Assume quarks to be infinitely heavy $\rightarrow$ only gluons.
Path integral: "Sum over all possible paths"

$$Z[J] = \int [dA] e^{-\int dx (\mathcal{L}_{YM} + A_\mu(x) J_\mu(x))}$$

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Path integral: "Sum over all possible paths"

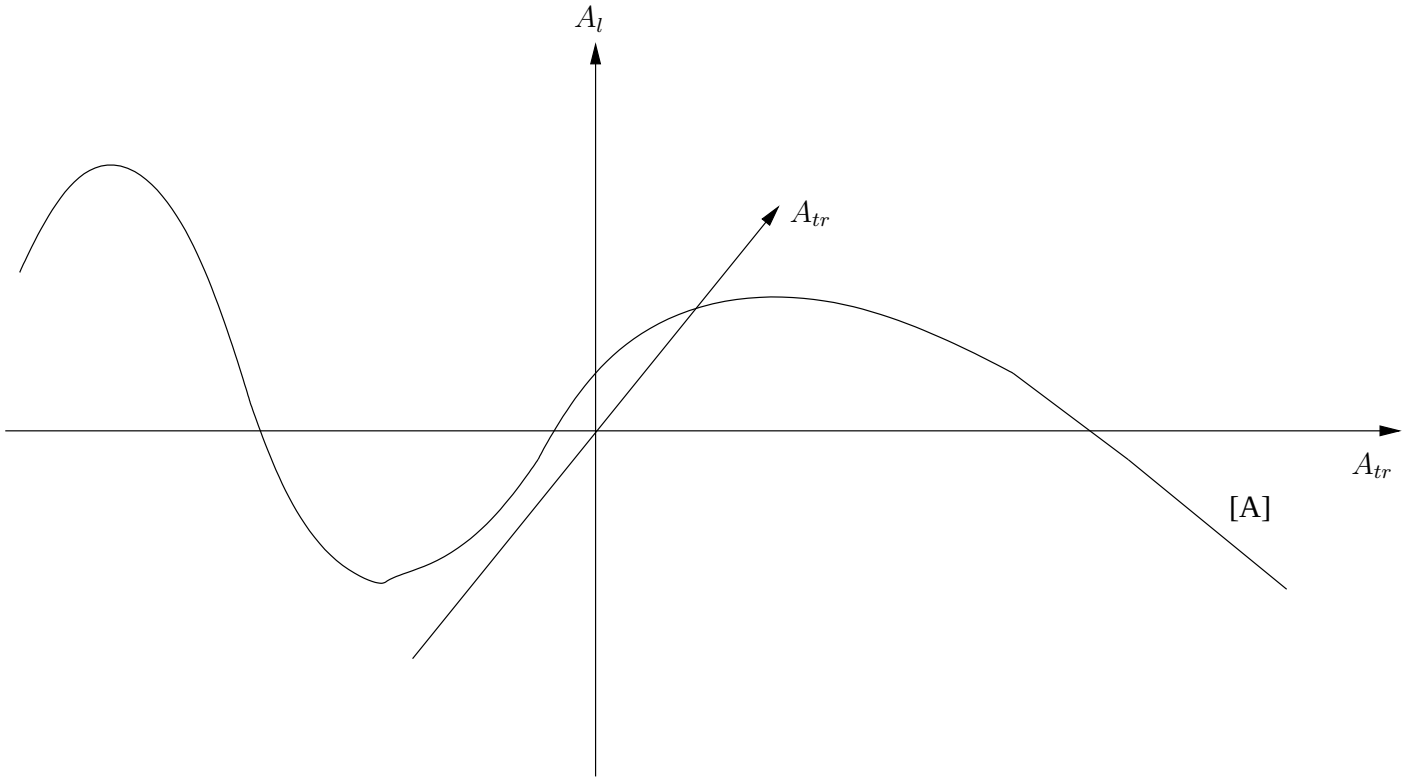
$$Z[J] = \int [dA] e^{-\int dx (\mathcal{L}_{YM} + A_\mu(x) J_\mu(x))}$$

The action $\int dx \mathcal{L}$ is invariant under gauge transformations = rotations in color space.
$\Rightarrow$ Equivalent configurations, $\mathcal{L}(A) = \mathcal{L}(A')$, exist (gauge copies), but functional integration $\int [dA]$ should only count one representative of a set of gauge copies $\Rightarrow$ gauge fixing.

[Other problems without gauge fixing: propagator not properly defined, commutation relations of the field operators cannot be obeyed.]

Pictorial sketch of gauge fixing

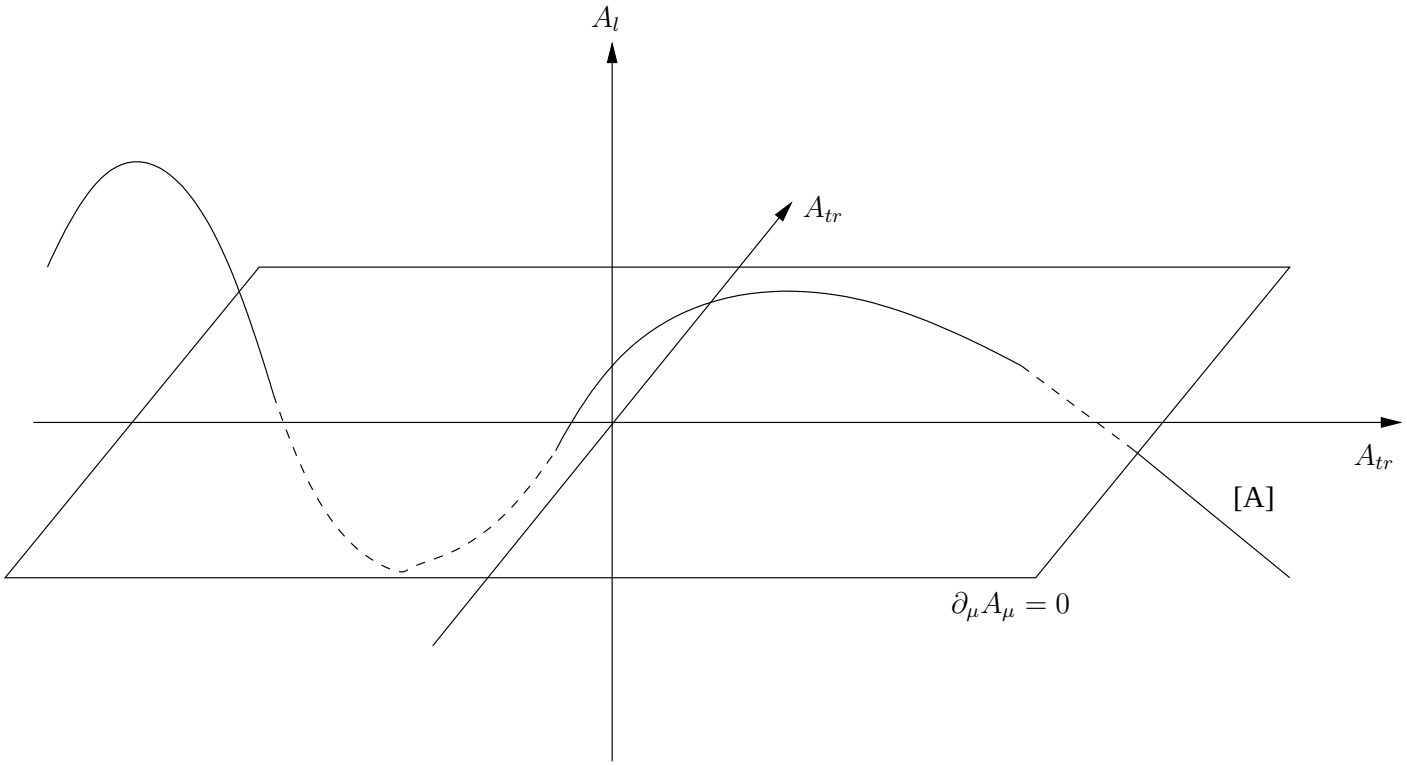
Sketch of field configuration space:



Configurations connected by a gauge transformation lie on a gauge orbit $[A]$.

Pictorial sketch of gauge fixing

Sketch of field configuration space:



Restrict integration the hyperplane $\partial_\mu A_\mu = 0$ (Landau gauge).

[Still copies $\rightarrow$ Gribov copies]

Gauge fixing

Task: Functional integration $\int [dA]$ should only count one representative of a set of gauge copies.

$\Rightarrow$ Restriction to a hyperplane in configuration space, e. g. $\partial_\mu A_\mu = 0$.

Gauge fixing

Task: Functional integration $\int [dA]$ should only count one representative of a set of gauge copies.

$\Rightarrow$ Restriction to a hyperplane in configuration space, e. g. $\partial_\mu A_\mu = 0$.

Convenient to introduce new Grassmann fields c and $\bar{c}$ to account for this restriction:

$$\Rightarrow Z[J, J_c, J_{\bar{c}}] = \int [dA\bar{c}c] e^{-\int dx \left(\mathcal{L} + \frac{1}{2\alpha}(\partial_\mu A_\mu)^2 + \boxed{\bar{c}(-\partial_\mu D_\mu)c} + AJ + \bar{c}J_{\bar{c}} + J_c c \right)}$$

[Hyperplane does not yield a unique solution $\rightarrow$ Gribov copies.]

The Yang-Mills Lagrangian in Landau gauge

Including the new terms, we have the following building blocks of the Lagrangian:

- propagators: AA, cc
- three-point interactions: AAA, Acc
- four-point interactions: $AAAA$

Gauge fixing terms

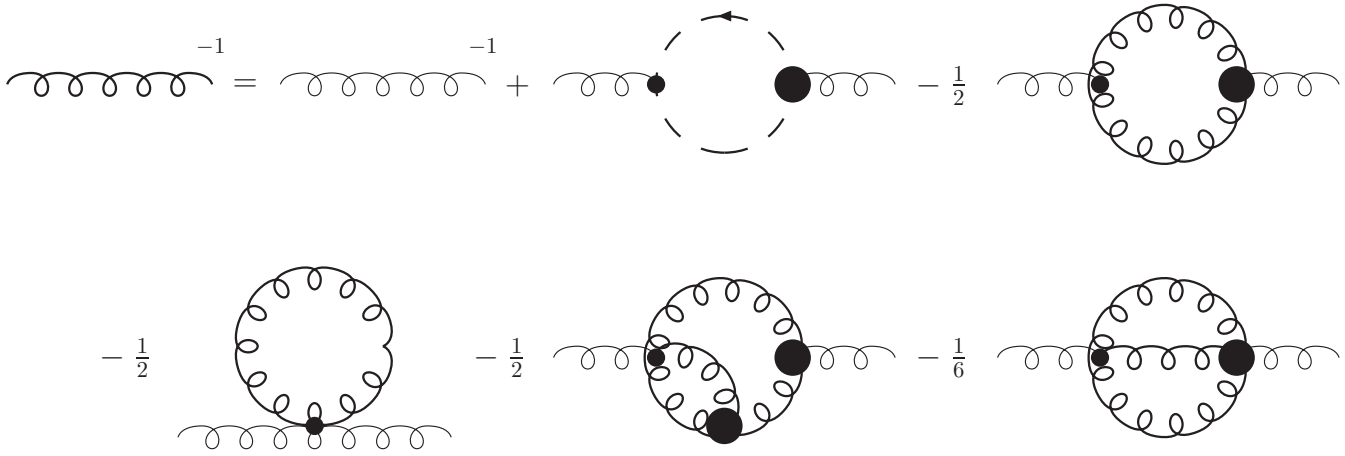
- Added the new ghost fields $\boldsymbol{c}$, including one interaction with the gluons.
- $\Rightarrow$ Modified AA propagator.

The Dyson-Schwinger equations of Landau gauge Yang-Mills theory

On a diagrammatic level one can "guess" the equations:

Building blocks: AA, cc; Acc, AAA, $AAAA$

One-loop terms: ghost-loop, gluon-loop, tadpole
Two-loop: squint, sunset



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Confinement scenarios

Several scenarios that describe confinement:

- Gribov-Zwanziger (Coulomb, Landau gauge)
  gluon propagator vanishing in the deep IR, ghost propagator IR divergent
- Kugo-Ojima
  Landau gauge: gluon propagator suppressed in the deep IR, ghost propagator IR divergent

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Generic propagator

$$L_{(\mu\nu)} \cdot \frac{D(p)}{p^2},$$

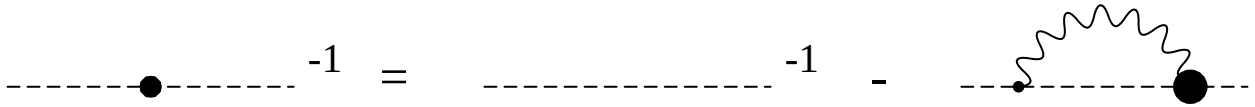
assume power law behavior at low p

$$D^{IR}(p) = A \cdot (p^2)^{\delta} \longleftarrow \text{IR exponent}$$

- $\delta < 0 \rightarrow$ IR enhancement
- $\delta > 1/2 \rightarrow$ IR vanishing
- Gribov-Zwanziger: $\delta_{gl} > 1/2$, $\delta_{gh} < 0$
- Kugo-Ojima: $\delta_{gl} \geq 1/2$, $\delta_{gh} < 0$

Power counting

- The ghost propagator DSE:



- Plug in power law ansätze for dressing functions in the IR

$$\left(\frac{B \cdot (p^2)^\beta}{p^2} \right)^{-1} \sim \int \frac{d^d q}{(2\pi)^d} \, P_{\mu\nu} \, \frac{A \cdot (q^2)^\alpha}{q^2} \, \frac{B \cdot ((p-q)^2)^\beta}{(p-q)^2} \, (p-q)_\mu q_\nu$$

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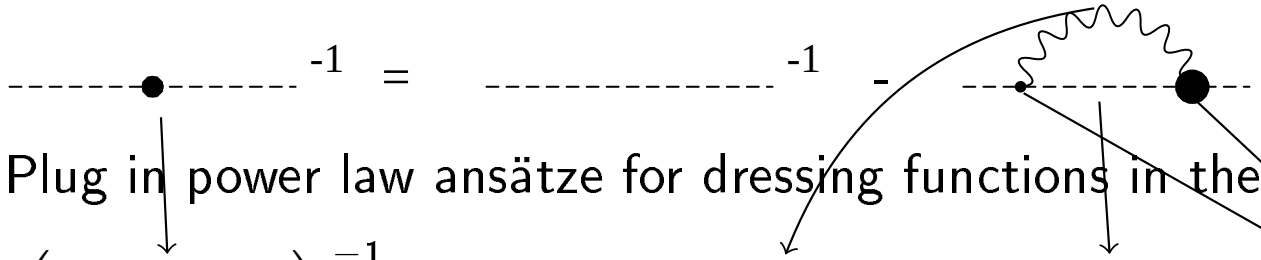

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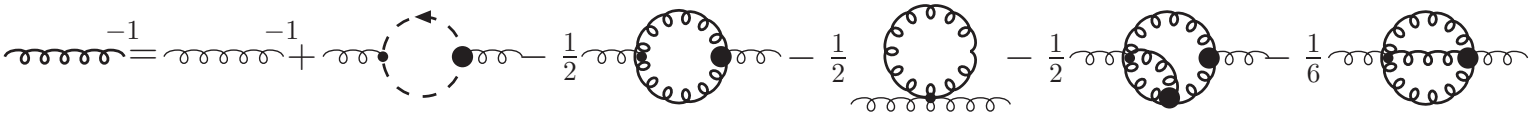
- Only one momentum scale
  $\longrightarrow$ simple power counting is possible:
- $1-\beta = \frac{d}{2} + \alpha - 1 + \beta - 1 + \frac{1}{2} + \frac{1}{2} \Longrightarrow -2\beta = \alpha + \frac{d}{2} - 2$

Constraints from DSEs

Still, at the end there is a big system of inequalities that has to be solved, e.g. gluon propagator in Landau gauge:

$$-\delta_{gl} = \min(\underbrace{0}_{\text{bare prop.}}, \underbrace{2\delta_{gl} + \delta 3g}_{\text{gh loop}}, \underbrace{2\delta_{gh} + \delta_{gg}}_{\text{gl loop}}, \underbrace{\delta_{gl}}_{\text{tadpole}}, \underbrace{4\delta_{gl} + 2\delta_{3g}}_{\text{squint}}, \underbrace{3\delta_{gl} + \delta_{4g}}_{\text{sunset}})$$



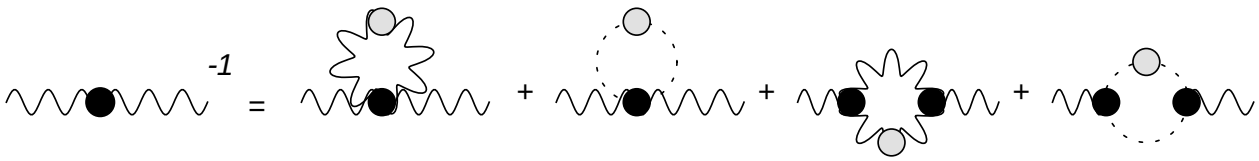
min-function ⇒ set of inequalities,
e.g. from sunset $4\delta_{gl} + \delta_{4g} \geq 0$.
From four-gluon DSE:
$\delta_{4g} \leq 0 \Rightarrow \delta_{gl} \geq 0$.

Functional renormalization group

Functional equations similar to DSEs, but with decisive differences:

- only 1-loop diagrams
- all quantities dressed

Renormalization group equations (RGEs) are "differential DSEs".



IR exponent for an arbitrary diagram

System difficult to solve. Look for alternatives or shortcuts.

Arbitrary Diagram v

Numbers of vertices and propagators related $\Rightarrow$ possible to get a formula for the IR exponent.

Function of:

- propagator IR exponents δ_{X_i}
- number of external legs m^{X_i}
- number of vertices.

$$\delta_v = -\frac{1}{2}\sum_i m^{X_i}\delta_{X_i} + \\ + \sum_i (\#\text{ of dressed vertices})_i C_1^i + \sum_i (\#\text{ of bare vertices})_i C_2^i$$

Restrictions

Combining DSEs and RGEs [cf. Fischer, Pawlowski, PRD 75 (2006)]

 +  = 

we get bounds for coefficients C_1^i and C_2^i:

$$C_1^i \geq 0, \qquad C_2^i \geq 0.$$

$\Rightarrow$ IR solution:

$$\delta_v = -\frac{1}{2}\sum_i m^{x_i}\delta_{x_i} + \sum_i (\#\text{ dr. vert.})_i C_1^i + \sum_i (\#\text{ bare vert.})_i C_2^i.$$

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$\Rightarrow$ Maximally IR divergent solution:

$$\boxed{\delta_{v,max} = -\frac{1}{2}\sum_i m^{x_i}\delta_{x_i}} + \cancel{\sum_i (\#\text{ dr. vert.})_i C_1^i} + \cancel{\sum_i (\#\text{ bare vert.})_i C_2^i}\ .$$

Only depends on number and type of external fields.

Existence of scaling solutions

Abundance of inequalities $\rightarrow$ reduce to relevant ones. Only small number left, e. g. Landau gauge:

$$\delta_{gh} + \frac{1}{2}\delta_{gl} \geq 0, \quad \delta_{gl} \geq 0.$$

Looking for solutions of the propagator DSEs $\rightarrow$ at least one of these equations has to be saturated:

- $\delta_{gl} = 0$: corresponds to trivial solution (perturbation theory)
- $2\delta_{gh} = -\delta_{gl}$: scaling solution for Landau gauge [von Smekal, Hauck, Alkofer, PRL (1997)]

Condition of saturation of one inequality: Restricts naive existence of scaling solutions in other gauges, e. g. covariant gauges, ghost anti-ghost symmetric gauges.

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Another possible mechanism for confinement

Dual superconductor scenario (electric $\leftrightarrow$ magnetic components)

Magnetic monopoles condense and the electric flux is squeezed into vortices ("strings").

Classical configurations of monopoles are constructed within the maximal Abelian subalgebra (Cartan subalgebra), i.e. its generators commutate:

$$[T^i, T^j] = 0.$$

Hypothesis of Abelian Dominance [Ezawa, Iwazaki, PRD 25 (1981)]

$\Rightarrow$ Abelian part should dominate in the infrared part of the theory.

Use maximally Abelian gauge (MAG).

The maximally Abelian gauge (MAG) in $SU(2)$

Split the gluon field $A_\mu = A^r_\mu T^r$
into diagonal and off-diagonal fields:

$$A_\mu = \boldsymbol{B^a T^a} + \boldsymbol{A^i T^i}$$

$\boldsymbol{T^a}$: off-diagonal matrices
$\boldsymbol{T^i}$: diagonal matrices
$\Rightarrow$ Gluonic vertices split: $\boldsymbol{ABB}$, $\boldsymbol{AABB}$, $\boldsymbol{BBBB}$
Number of diagonal fields per vertex restricted.

Non-linear gauge fixing

Minimize off-diagonal components along gauge orbit:

$$D^{ab}_\mu \boldsymbol{B^b_\mu} = (\delta_{ab}\partial_\mu - g\, f^{abi} \boldsymbol{A^i_\mu}) \boldsymbol{B^b_\mu} = 0$$

$\Rightarrow$ ghosts and interactions: $\boldsymbol{Acc}$, $\boldsymbol{AAcc}$, $\boldsymbol{BBcc}$

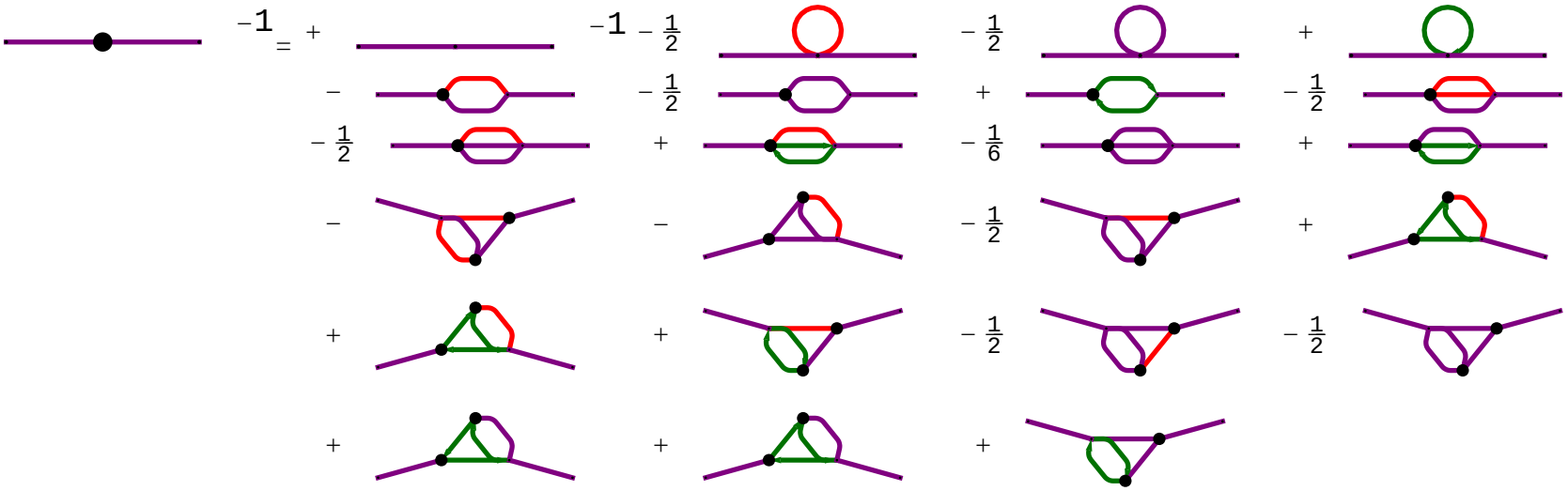
builing blocks for DSEs: $\boldsymbol{ABB}$, $\boldsymbol{Acc}$, $\boldsymbol{AABB}$, $\boldsymbol{AAcc}$, $\boldsymbol{BBcc}$, $\boldsymbol{BBBB}$, $\boldsymbol{cccc}$

DSEs of the MAG

diagonal gluon: off-diagonal gluon:

off-diagonal ghost:

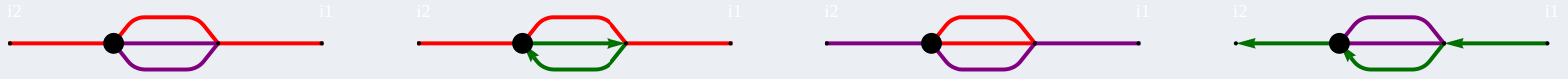
off-diagonal gluon propagator DSE:



Results for the MAG in $SU(2)$

Leading diagrams in propagator DSEs

All sunsets with 2 diagonal & 2 off-diagonal fields



possibly some squints

- IR enhanced diagonal gluon:

$$\delta_B = \delta_c =: \kappa \geq 0, \qquad \delta_A = -\kappa$$

- Supports Abelian dominance hypothesis (relevant degrees of freedom in diagonal part of gluon field).
- Vertices with 2 diagonal & 2 off-diagonal fields do NOT scale, i.e. $\delta_{AABB} = \delta_{AAcc} = 0$.

The MAG in $SU(N)$

In general there are more interactions than included above.
$\rightarrow$ Different solution for "physical system", i. e. $SU(3)$?

4 additional vertices: ***BBB***, ***Bcc***, ***ABBB***, ***ABcc***
Constraints:

$$\frac{3}{2}\delta_B \geq 0, \qquad \frac{1}{2}\delta_B + \delta_c \geq 0,$$
$$\frac{1}{2}\delta_A + \frac{3}{2}\delta_B \geq 0, \qquad \frac{1}{2}\delta_A + \frac{1}{2}\delta_B + \delta_c \geq 0$$

Already contained in "old" system $\rightarrow$ nothing new, solution still valid and unique.

Summary: Functional approaches & scaling solutions

Functional approaches

- Gauge fixing necessary for functional approaches.
- Infinitely large tower of equations.
- Solution in the IR possible without truncations.

Scaling solutions

- Power laws for dressing functions with infrared exponents in IR regime.
- Maximally IR divergent solution depends on number of legs and propagator IR exponents.
- At least one vertex does not scale.

Summary: Results for the MAG

First full analytic IR analysis of MAG without truncations:

- Diagonal gluon ($\boldsymbol{A}$) IR enhanced
  $\Rightarrow$ supports Abelian dominance hypothesis.
- No qualitative difference between $SU(2)$ and $SU(3)$.

However: Previous results (lattice: Mendes, Cucchieri; extended Gribov-Zwanziger framework: Sorella et al.) with IR finite propagators.

It may be possible that these two solutions coexist (cf. Landau gauge) $\longrightarrow$ new solution with nice properties like IR divergent quantities found.

The presented IR solution provides vital information for a full numerical solution.