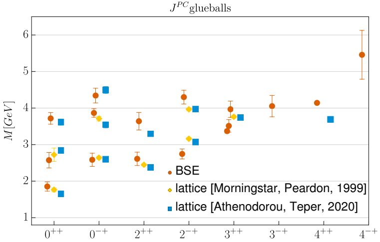
# Bound states in strong interaction physics (from a functional point of view)

$$\mathcal{L}_{\text{QCD}} = \overline{\psi}(-\not{D} + m)\psi + \mathcal{L}_{YM} \qquad \rightarrow$$





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Physik-Combo (RTG 2522 Jena Leipzig)

Jena, Germany

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# Content and what to expect

- Bound states and quantum chromodynamics

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- Personal, biased selection of examples
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- No pentaquarks, hybrids (status: exploratory)
- Focus on spectrum $\rightarrow$ no form factors etc.
- Challenges of functional bound state calculations

# Reading material

This presentation (with links): mqh.at/physics/presentations

A small selection to get started:

- R. Alkofer and L. von Smekal, “The Infrared behavior of QCD Green’s functions: Confinement dynamical symmetry breaking, and hadrons as relativistic bound states”, Phys. Rept. 353 (2001) 281, hep-ph/0007355
- P. Maris and C. D. Roberts, “Dyson-Schwinger equations: A Tool for hadron physics”, Int. J. Mod. Phys. E 12 (2003) 297, nucl-th/0301049
- A. Bashir, L. Chang, I. C. Cloet, B. El-Bennich,Y. X. Liu, C. D. Roberts and P. C. Tandy, “Collective perspective on advances in Dyson-Schwinger Equation QCD”, Commun. Theor. Phys. 58 (2012) 79, arXiv:1201.3366
- Baryons: G. Eichmann, H. Sanchis-Alepuz, R. Williams, R. Alkofer, C.S. Fischer, “Baryons as relativistic three-quark bound states”, Progress in Particle and Nuclear Physics 91 (2016) 1-100, arXiv:1606.09602
- Christian S. Fischer, Hadron physics with functional methods, Internationale Universtitätswochen für Theoretische Physik, Admont, 2017

# Reading material, cont.

Special topics:

- Tetraquarks: G. Eichmann, C. S. Fischer, W. Heupel, N. Santowsky and P. C. Wallbott, “Four-Quark States from Functional Methods”, Few Body Syst. 61 (2020) no.4, 38, arXiv:2008.10240
- Glueballs: M. Q. Huber, C. S. Fischer and H. Sanchis-Alepuz, “Higher spin glueballs from functional methods”, Eur. Phys. J. C 81 (2021) no.12, 1083, arXiv:2110.09180
- Correlation functions: M. Q. Huber, “Nonperturbative properties of Yang-Mills theories”, Phys. Rept. 879 (2020) 1, arXiv:1808.05227; M. Q. Huber, “Correlation functions of Landau gauge Yang-Mills theory”, Phys. Rev. D 101 (2020), 114009, arXiv:2003.13703

If you want to know (technical) details:

- Derivation of correlation functions: M. Q. Huber, A. K. Cyrol and J. M. Pawlowski, “DoFun 3.0: Functional equations in Mathematica”, Comput. Phys. Commun. 248 (2020), 107058, arXiv:1908.02760
- Technical basics: see webpage (material from Doctoral Training Program 2022, ECT*, Trento)
- Advanced techniques: H. Sanchis-Alepuz and R. Williams, “Recent developments in bound-state calculations using the Dyson–Schwinger and Bethe–Salpeter equations”, Comput. Phys. Commun. 232 (2018), 1-21, arXiv:1710.04903

# Bound states

Constituents bound by some force.

- Localized
- Attractive force
- Behaves as a single object (under certain conditions)
- Discrete spectrum (as opposed to free constituents)
- 2 or more constituents

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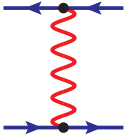

2 fermions in QED:

- Example: Hydrogen atom
- one-photon exchange
- Coulomb potential $\propto 1/r$
- spin-orbit coupling: fine splitting
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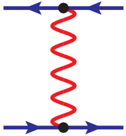

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We will look for poles in *n*-point functions/scattering matrices!

# The strong interaction

Quantum chromodynamics: gauge theory

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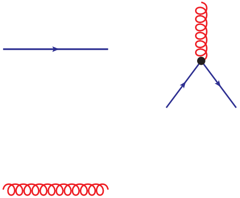


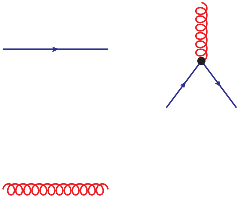
$$\mathcal{L}_{\text{QED}} = \bar{\psi}\,(-\not{D} + m)\psi$$

$$+\frac{1}{2}\,\text{Tr}\{F_{\mu\nu}F^{\mu\nu}\}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

# The strong interaction

Quantum chromodynamics: non-Abelian gauge theory



$$\mathcal{L}_{\boxed{\text{QCD}}} = \sum_{\text{flavor } f} \overline{\psi}_f(-\not{D} + m)\psi_f + \frac{1}{2}\,\text{Tr}\{F_{\mu\nu}F^{\mu\nu}\}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\,g\,[A_\mu, A_\nu]$$

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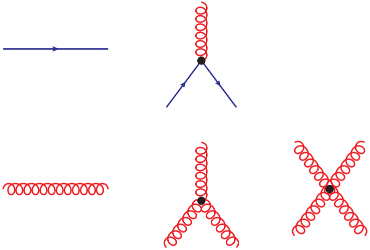

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# Properties of the strong interaction

- Confinement: “no free quarks or gluons”
  dual superconductor picture, center vortices, Kugo-Ojima, ... many open questions

- Dynamical mass creation:
  - light quarks $\sim$ *MeV*
  - proton $\sim$ *GeV*
  - $\rightarrow$ chiral symmetry and its breaking

- Rich spectrum: mesons, baryons, exotics (XYZ states, multiquark states, states with gluonic content)

# From protons to quarks

Status 1947: Electron, proton, neutron, photon
$\rightarrow$ Build the world around us.

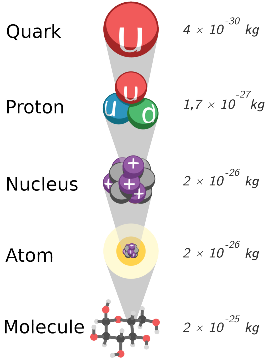
Cosmic rays: positron, pions, muon (Rabi: “Who ordered that?”)
hypothesized: neutrino

1947-1950: Kaons, Lambda

“Particle zoo”: Many new particles (hadrons) found
Pauli: “Had I foreseen that, I would have gone into botany.”

Quark model: 1964, Gell-Mann, Zweig, hadrons are composite of quarks

Deep inelastic scattering experiments 1968: point-like particles inside protons

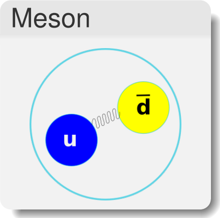


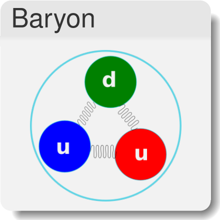

Alexander Gorfer (quant.uni-graz.at), CC-BY-SA 4.0, mod.

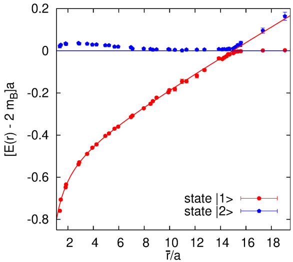
# Bound states of the strong interaction

Quark model 1964:

- Solve Schrödinger equation with a given potential, e.g., Cornell:
  $V(r) = -\frac{4}{3}\frac{\alpha_s}{r} + \sigma r + \text{const.}$
- Abundance of states




- Yang-Mills (infinitely heavy quarks): potential rises linearly
- QCD: string between quarks can break $\rightarrow$ creation of quark/antiquark pair

[Bali et al., Phys. Rev. D 71 (2005)]

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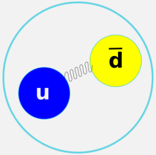
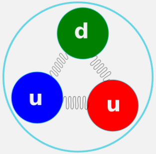
**Meson**

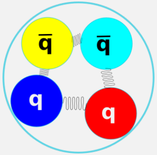



**Baryon**

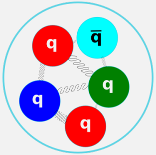



Exotics:

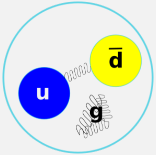
**Tetraquark**




**Pentaquark**




**Hybrid**



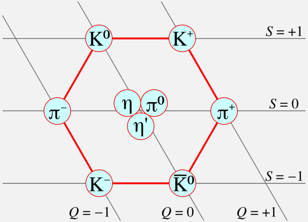

**Glueball**

# Multiplets

## Quark model

Classification in terms of mesons or baryons $\rightarrow$ multiplets

Outside this classification $\rightarrow$ exotics

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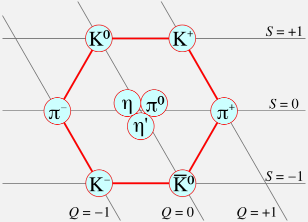


$J^{PC} = 0^{-+}$

Classification not always easy, e.g., scalar sector $J^{PC} = 0^{++}$. $\rightarrow$ tetraquarks, glueballs

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Hadron masses from correlation functions of color singlet operators.

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Lattice: Mass from exponential Euclidean time decay

$$\lim_{t\to\infty}\langle O(x)O(0)\rangle \sim e^{-t\,M}$$

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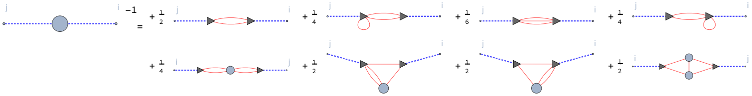
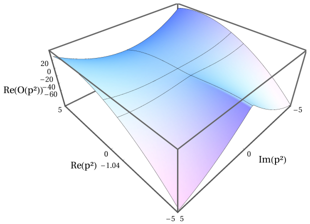
Functional approach:
Glueball:



+ 3-loop diagrams [MQH, Cyrol, Pawlowski, Comput.Phys.Commun. 248 (2020)]




Leading order: [Windisch, MQH, Alkofer, Phys.Rev.D87 (2013)]

# Derivation of bound state equations I

For more details see [Eichmann, Sanchis-Alepuz, Williams, Alkofer, Fischer, Prog.Part.Nucl.Phys. 91 (2016)]

**$n$-particle bound state**

Pole in a $2n$-point function.

For simplicity here $n = 2$.

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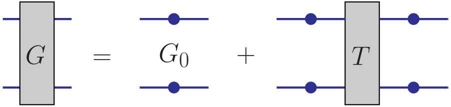
Full 4-point function:

$$G = G_0 + G_0\, T\, G_0$$




disconn.    connected part

$\rightarrow$ scattering matrix $T$ (amputated, conn. part of $G$)

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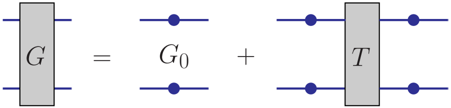


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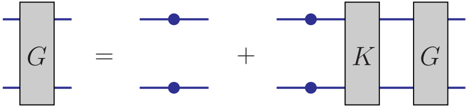
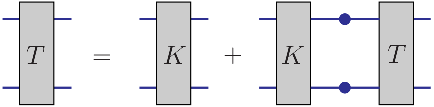
Dyson equations: nonperturbative resummations! Compare:

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + \ldots = 1 + x\, f(x) = 1 + x + x^2\, f(x)$$

$$G = G_0 + G_0\, K\, G$$




$$T = K + K\, G_0\, T$$

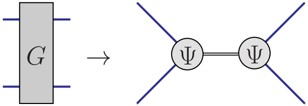



Scattering kernel $K$: 2-particle irreducible with respect to horizontal quarks lines (created by iteration)

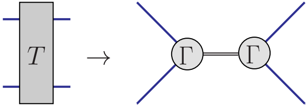
# Derivation of bound state equations II

On-shell: at pole position $P^2 = -M^2$

$$G \to \frac{\Psi\,\overline{\Psi}}{P^2 + M^2}$$




$$T \to \frac{\Gamma\,\overline{\Gamma}}{P^2 + M^2}$$




Bethe-Salpeter amplitude $\Gamma$ is the amputated wave function, $\Psi = G_0\Gamma$.

Pole position: mass $M$

Residues: $\Gamma\,\overline{\Gamma}$, $\Psi\,\overline{\Psi}$

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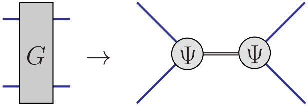


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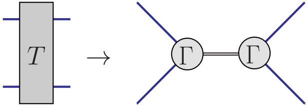


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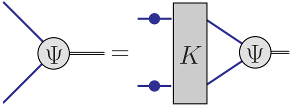
Plug into Dyson equations: homogeneous Bethe-Salpeter equations

$$\Psi = G_0\, K\, \Psi$$

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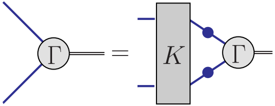


# Elements of a BSE

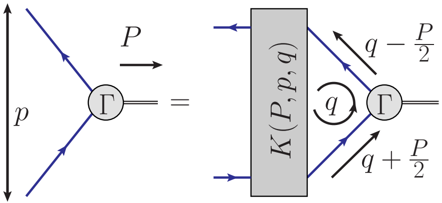
$$\Gamma = K\, G_0\, \Gamma$$

Input:

- Propagators $G_0$
- Kernel $K$

Output:

- Mass $M$
- Bethe-Salpeter amplitudes $\Gamma$




(quark-antiquark state)

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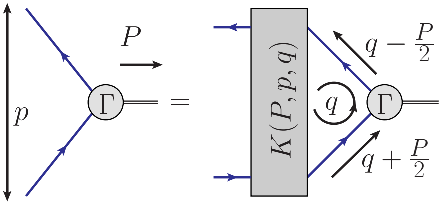



(quark-antiquark state)

Symmetry constraints: Propagators and kernels are not independent!

Relevant for QCD: Chiral symmetry in quark sector $\rightarrow$ axial-vector Ward-Takahashi identity

# Functional elements

**Central object**

1PI effective action $\Gamma[\Phi]$

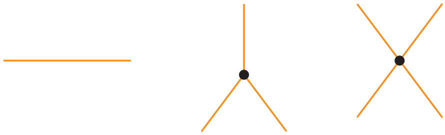
$\Gamma[\Phi]$ is the generating functional of 1PI correlation functions. $\rightarrow$ Vertex expansion:

$$\Gamma[\Phi] = \sum_{i=0}^{\infty} \frac{1}{\mathcal{N}^{i_1 \ldots i_n}} \sum_{i_1, \ldots, i_n} \underset{\substack{\uparrow \\ \text{vertices}}}{\Gamma^{i_1 \ldots i_n}} \Phi_{i_1} \ldots \Phi_{i_n}$$

# Generating functionals

Example: Scalar theory (Keep things simple...)

$$S[\phi] = \int dx \left( \phi(-\partial^2 + m^2)\phi + \frac{\lambda_3}{3!}\phi^3 + \frac{\lambda_4}{4!}\phi^4 \right)$$



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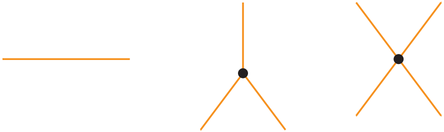
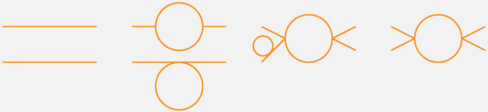


Path integral:

$$Z[J] = \int D[\phi] e^{-S[\phi] + \int dx \phi(x) J(x)} = e^{W[J]}$$

$Z[J] \rightarrow$ Generating functional for full correlation functions



$W[J] \rightarrow$ Generating functional for connected correlation functions

# 1PI effective action

Legendre transform: New variable $\Phi(x)$ (averaged field $\Phi$ in presence of external source $J$)

$$\boldsymbol{\Phi(x)} := \langle\phi(x)\rangle_J = \frac{\delta W[J]}{\delta J(x)} = Z[J]^{-1}\int D[\phi]\boldsymbol{\phi(x)}e^{-S[\phi]+\int dy\phi(y)J(y)} \qquad \left(J(x) = \frac{\delta\Gamma[\Phi]}{\delta\Phi(x)}\right)$$

$$\Gamma[\boldsymbol{\Phi}] = -W[J] + \int dx\boldsymbol{\Phi(x)}J(x)$$

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$\Gamma[\Phi] \to$ 1PI effective action, generating functional of one-particle irreducible correlation functions

(All correlation functions can be constructed from 1PI correlation functions.)

# Propagators and vertices

Propagator:

$$D(x,y) = D(x-y) = \left.\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)}\right|_{J=0} = \langle\phi(x)\phi(y)\rangle - \langle\phi(x)\rangle\langle\phi(y)\rangle$$

$$D(x,y)^J := \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} = \left(\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(x)\delta\Phi(y)}\right)^{-1}$$

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Derivatives of 1PI effective action: (Note $J \neq 0$ and "$-$" by convention.)

$$\Gamma(x_1,\ldots,x_n)^J := -\frac{\delta\Gamma[\Phi]}{\delta\Phi(x_1)\cdots\delta\Phi(x_n)}$$

**Physical vertices**

$$\Gamma(x_1,\ldots,x_n) := \Gamma(x_1,\ldots,x_n)^{J=0}, \quad n > 2$$

# Derivation of DSEs

Details in appendix

Integral of a total derivative vanishes:

$$0 = \int D[\phi] \frac{\delta}{\delta\phi} e^{-S+\int dy\phi(y)J(y)} = \int D[\phi] \left( -\frac{\delta S}{\delta\phi(x)} + J(x) \right) e^{-S+\int dy\phi(y)J(y)}$$

# Derivation of DSEs

Details in appendix

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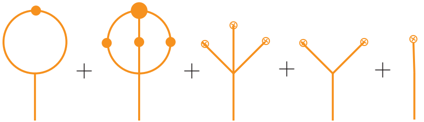
## Master DSE for 1PI correlation functions

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} = \frac{\delta S}{\delta\phi(x)}\bigg|_{\phi(x')=\Phi(x')+\int dz\, D(x',z)^J \delta/\delta\Phi(z)}$$

Get DSE for $n$-point function by applying $n-1$ derivatives.

$$\frac{\delta}{\delta\phi}\Gamma[\phi] =$$



# Automatized derivation with *DoFun*

**Derivation of functional equations** [Alkofer, MQH, Schwenzer, ’08; MQH, Braun, ’11; MQH, Cyrol, Pawlowski, ’19]

→ `https://github.com/markusqh/DoFun/`

Works in two steps:

- Symbolic derivation (no Feynman rules, just types of fields)
- Algebraic: Plug in Feynman rules

See also *QMeS-Derivation*
[Pawlowski, Schneider, Wink, CPC 287 (2023)]
→ `https://github.com/QMeS-toolbox/`

## doDSE

*doDSE*[*ac*, *flis*, [*opts*]] derives the DSE from the action *ac* for the fields contained in *flis*.

*doDSE*[*ac*, *flis*, *props*, [*opts*]] derives the DSE only with propagators contained in prop

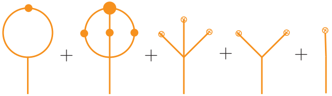
*doDSE*[*ac*, *flist*, *vtest*, [*opts*]] derives the DSE only with vertices allowed by *vtest*.
Allowed propagators will be taken from *ac* if the *props* argument is not given.

▾ Details

- The following options can be given:

| | |
|---|---|
| sourcesZero | True |

# DSEs $\leftrightarrow$ flow equations

| Dyson-Schwinger equations (DSEs) | Functional RG equations (FRGEs) |
|---|---|
| 'integrated flow equations' | 'differential DSEs' |
| effective action $\Gamma[\phi]$ | effective average action $\Gamma^k[\phi]$ |
| – | regulator |
| n-loop structure (n *const*.) | 1-loop structure |
| exactly only bare vertex per diagram | no bare vertices |
| $\frac{\delta}{\delta\phi}\Gamma[\phi] =$  | $k\frac{\partial}{\partial k}\Gamma^k[\phi] =$  |

- Both systems of equations are exact.
- Both contain infinitely many equations.

# Quark propagator

Described by 2 dressing functions:

$$\left(S^{ij}\right)^{-1} = \delta^{ij}\left(i\,\not{p}\,A(p^2) + B(p^2)\right)$$

$$S^{ij} = \delta^{ij}\,\frac{-i\,\not{p}\,A(p^2) + B(p^2)}{p^2\,A(p^2)^2 + B(p^2)^2}$$

$$= \delta^{ij}\,\frac{\boldsymbol{Z_f(p^2)}}{p^2 + \boldsymbol{M(p^2)}^2}\left(-i\,\not{p} + \boldsymbol{M(p^2)}\right)$$

Quark renormalization function
$\boldsymbol{Z_f(p^2)} = 1/A(p^2)$

Quark mass function
$\boldsymbol{M(p^2)} = B(p^2)/A(p^2)$

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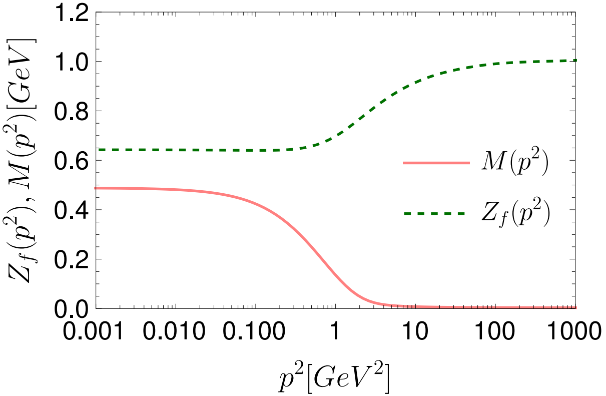


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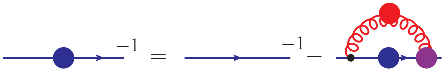
# Quark propagator Dyson-Schwinger equation

- (Use a model.)

# Quark propagator Dyson-Schwinger equation

- (Use a model.)
- Calculate the propagator.

Dyson-Schwinger equation (exact!):




$$S(p)^{-1} = S_0^{-1} - \Sigma(p),$$

$$\Sigma(p) = -C_F\, g^2 \int \frac{d^4q}{(2\pi)^4} \mathrm{tr}\{\gamma^\mu S(q) D_{\mu\nu}(k) \Gamma^\nu(-k; -p, q)\}$$

- Gluon propagator $\boldsymbol{D_{\mu\nu}(k)} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \boldsymbol{Z(k^2)}$
- Quark-gluon vertex $\boldsymbol{\Gamma^{a,\nu}(k; p, q)} = i\, g\, T^a \sum_{i=1}^{12} \tau_i^\mu(k; p, q)\, \boldsymbol{h_i(k; p, q)}$

# Approximations: Simple model input

Bare vertex: $\gamma_\mu$

Gluon propagator:

- Munczek-Nemirovsky model (local in momentum space):

  $D_{\mu\nu}(k) \propto \delta_{\mu\nu}\delta(k) \to$ algebraic equations

  Mass creation

- Nambu–Jona-Lasinio/contact model (local in position space):

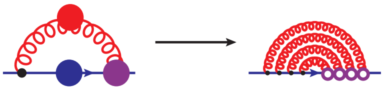
  $D_{\mu\nu}(k) \propto \delta_{\mu\nu} c/\Lambda^2 \to$ four-fermi interaction (cutoff as add. parameter)

Critical behavior in coupling $\to$ dynamical symmetry breaking

# Approximations: Rainbow

Need the gluon propagator ($\boldsymbol{Z(k^2)}$) and the quark-gluon vertex ($\boldsymbol{h_i(k;p,q)}$).

- $\boldsymbol{\Gamma^{a,\nu}(k;p,q)} \propto \gamma^\nu h_1(k;p,q)$
- $\dfrac{g^2}{4\pi} Z(k^2) h_1(k;p,q) \propto \boldsymbol{\alpha(k^2)}$



Iteration $\to$ only 'rainbow-like' diagrams

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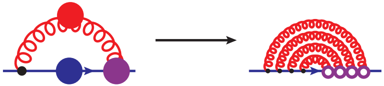

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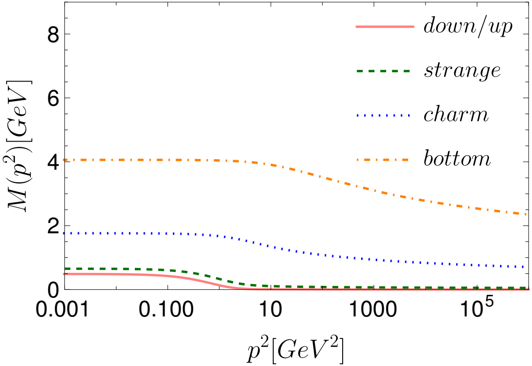
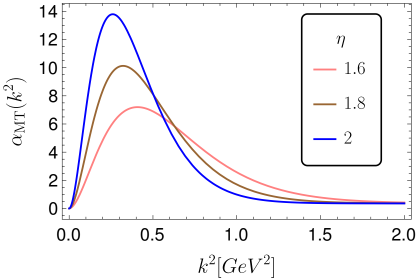


# Example for a model: Maris-Tandy interaction

[Maris, Roberts, Tandy, Phys. Rev. C 56 (1997); Maris, Tandy, Phys. Rev. C 60 (1999)]:

$$\alpha(k^2) = \underbrace{\pi\,\eta^7 \left(\frac{k^2}{\Lambda^2}\right)^2 e^{-\eta^2 \frac{k^2}{\Lambda^2}}}_{\alpha_{\text{IR}}(k^2)} + \alpha_{\text{UV}}(k^2)$$



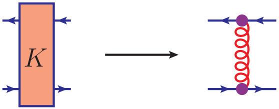

- Scale $\Lambda$ from $f_\pi$
- Quark masses $m_u = m_d$, $m_s$ from $m_\pi$, $m_K$
- Parameter $\eta$: window of small sensitivity (for meson masses and decay constants)
- $\alpha_{\text{UV}}$: Phenomenologically irrelevant, provides correct perturbative running to quark propagator

# Kernel approximations

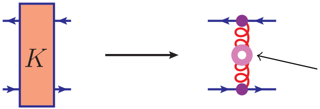
Kernel: “all interactions which are two-particle irreducbible with respect to two horizontal quark lines”

Examples:

- Pertubation theory: one-particle exchange
- Models
- Systematic derivation from effective actions (see glueballs)



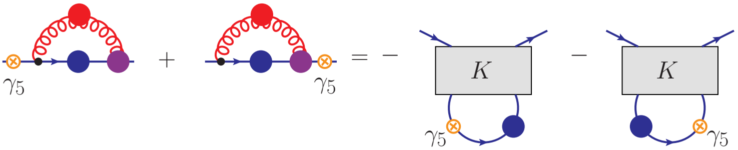

Analog to rainbow truncation:
ladder truncation



# Chiral symmetry

Massless QCD with 3 flavors: $U_V(1) \times SU_V(3) \times U_A(1) \times SU_A(3)$ flavor symmetry

Consequence of chiral symmetry for bound state equations:
Relation between quark selfenergy and kernel



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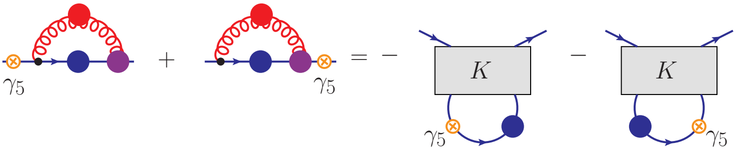


Chiral symmetry spontaneously broken $\rightarrow$ Goldstone theorem: massless bosons ($\pi$, $K$, $\eta$)

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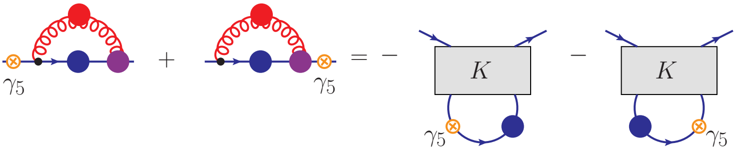


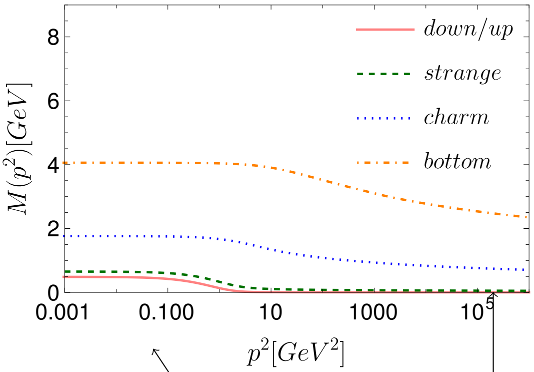

Chiral symmetry spontaneously broken $\rightarrow$ Goldstone theorem: massless bosons ($\pi$, $K$, $\eta$)

Explicitly broken by quark masses, but quark masses small. $\rightarrow$ Goldstone bosons are light.

$\rightarrow$ Nontrivial to fulfill!

- Rainbow-ladder: ✓
- Explicit construction for beyond rainbow-ladder, e.g., [Bender, Roberts, von Smekal, Phys.Lett.B 380 (1996); Williams, Fischer, Heupel, Phys. Rev. D 93 (2016); Qin, Roberts, Chin.Phys.Lett. 38 (2021)] $\rightarrow$ Cumbersome.

# Dynamic mass creation




dynamic mass creation in QCD

quark mass from Englert-Brout-Higgs effect

- Consequence of dynamical breaking of chiral symmetry.
  Order parameters: $M(0)$, chiral condensate $\langle\bar\psi\psi\rangle \sim \int dq\,\mathrm{Tr}\,S(q)$
- UV: quark mass as external parameter from QCD, “current quark mass“
- IR: created mass, ”constituent quark mass”
- Most (visible) mass is created by QCD and not the EBH effect!
- Proton: $\sim 940$ MeV, 3 light quarks $\sim 15$ MeV

# Amplitudes

$$\Gamma = K\, G_0\, \Gamma$$

$J^{PC}$ $\rightarrow$ Encoded in amplitude $\Gamma$:

$$\Gamma(P,p) = \sum_{i=1}^{n} \tau^i(P,p)\, h_i(P,p)$$

Quark-antiquark-state $\rightarrow$ Dirac indices
Spin $\rightarrow$ Lorentz indices

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Finite number of tensors $\tau_i$ compatible with given $J^{PC}$!

Quark-antiquark-state $\rightarrow$ Dirac indices
Spin $\rightarrow$ Lorentz indices

**Example: (pseudo)scalar mesons ($J^{PC} = 0^{\pm+}$)**

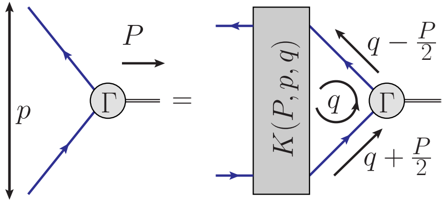
scalar ($P = +1$):

$$\tau^i(P,p) = \{\mathbb{1}, i\,\not{P}, i\,\not{p}, [\not{p}, \not{P}]\}$$

pseudoscalar ($P = -1$):

$$\tau^i(P,p)\gamma_5$$

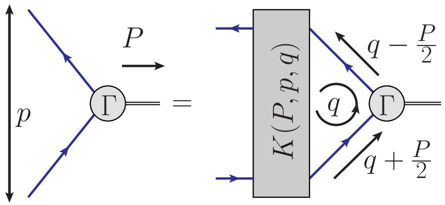
# Mass




$$\boldsymbol{\lambda(P)}\Gamma(P) = \mathcal{K} \cdot \Gamma(P)$$

$\rightarrow$ Eigenvalue problem for $\Gamma(P)$

# Mass

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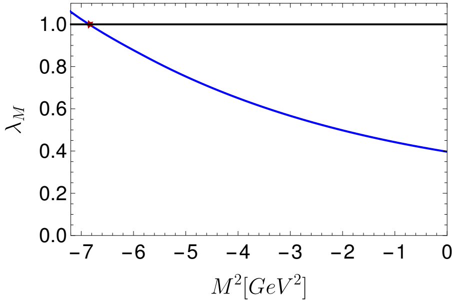
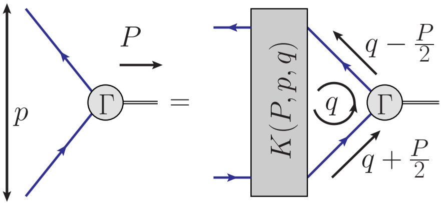



Find $P$ with $\lambda(P) = 1. \Rightarrow M^2 = -P^2$

# Calculation for $P^2 = -M^2$

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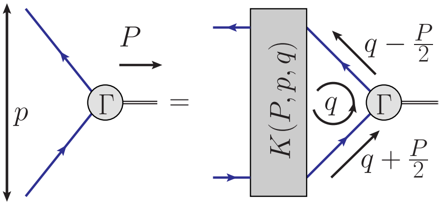


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$\rightarrow$ Eigenvalue problem for $\Gamma(P)$:
Find $P$ with $\lambda(P) = 1$.
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Propagators are probed at $\left(q \pm \frac{P}{2}\right)^2 = \frac{P^2}{4} + q^2 \pm \sqrt{P^2\, q^2} \cos\theta = -\frac{M^2}{4} + q^2 \pm \boldsymbol{i}\, M \sqrt{q^2} \cos\theta$

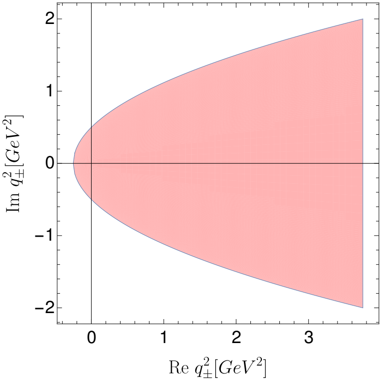
$\rightarrow$ Complex for $P^2 < 0$!

Time-like quantities ($P^2 < 0$) $\rightarrow$ Correlation functions for complex arguments.

# Quark propagator for complex arguments

Integration region ($M = 1\,\text{GeV}$):

$$\left(q \pm \frac{P}{2}\right)^2 = -\frac{M^2}{4} + q^2 \pm \boldsymbol{i}\, M \sqrt{q^2}\, \cos\theta$$



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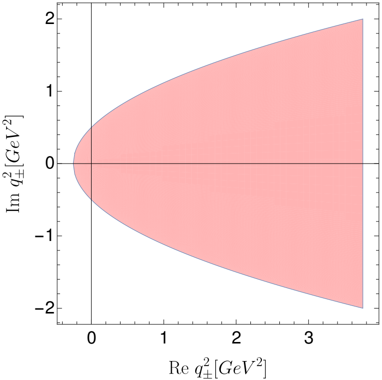
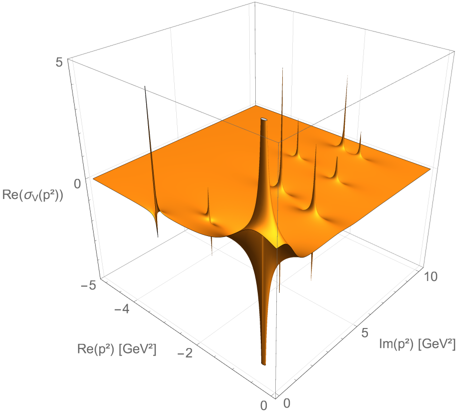



Analytic structure with Maris-Tandy model:




[Windisch, Phys. Rev. C 95 (2017)]

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$\Rightarrow$ Accessible $M$ determined by poles in propagator.

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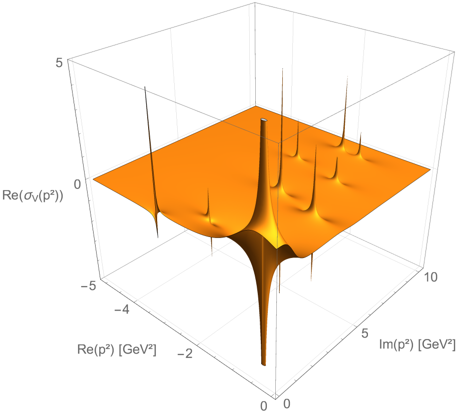


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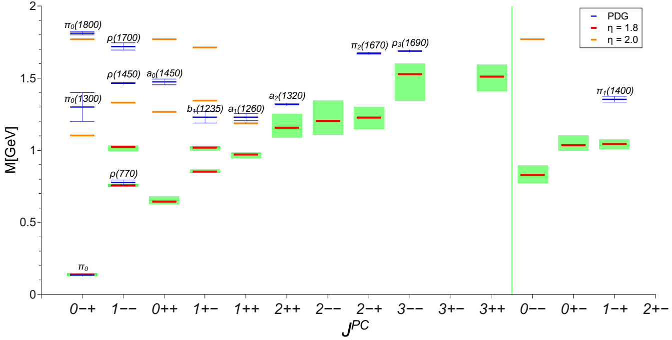
# Mesons from rainbow-ladder with Maris-Tandy interaction

$q\bar{q}$:




[Fischer, Kubrak, Williams, Eur.Phys.J.A50 (2014)]

- Well investigated for more than 20 years
- Describes pseudoscalar and vector ground states well
- Not so good for other quantum numbers
- Also 'exotic' quantum numbers

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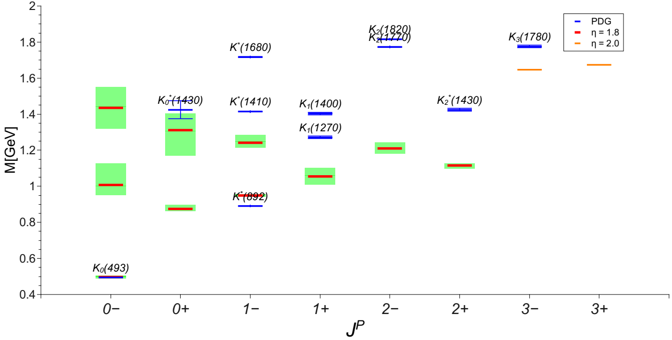


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# Baryons: 3-body bound state equation

[Eichmann, Sanchis-Alepuz, Williams, Alkofer, Fischer, Prog.Part.Nucl.Phys. 91 (2016)]

proton: *uud* quarks $\rightarrow$ three constituents ($u = d$: nucleon)

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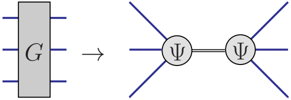
Three-body bound states in six-point functions. $\rightarrow$
Faddeev equation



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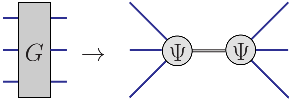


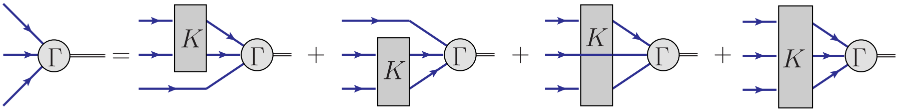


- 2- and 3-body interactions

# Baryons: 3-body bound state equation

[Eichmann, Sanchis-Alepuz, Williams, Alkofer, Fischer, Prog.Part.Nucl.Phys. 91 (2016)]

proton: *uud* quarks $\rightarrow$ three constituents ($u = d$: nucleon)

Three-body bound states in six-point functions. $\rightarrow$
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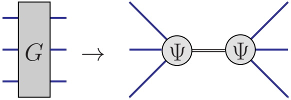


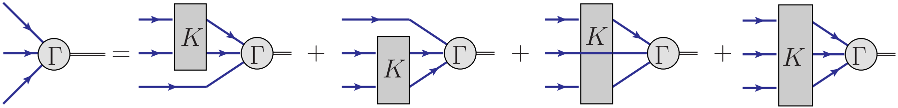


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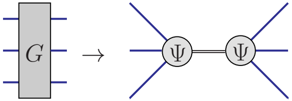
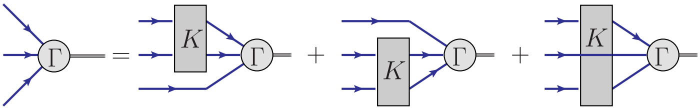
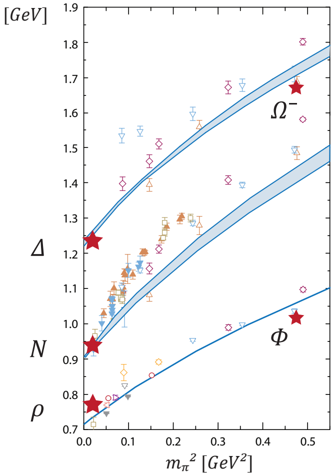



- 2- and 3-body interactions
- 3 momenta (1 total, 2 relative)
- Leading contribution (via three-gluon vertex) of 3-body interaction vanishes due to color

# Baryon masses

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**Approximation**

- Rainbow-ladder
- Maris-Tandy interaction

- First covariant 3-body calculation of nucleon *N*: [Eichmann, Alkofer, Krassnigg, Nicmorus, Phys. Rev. Lett. 104 (2010); Eichmann, Phys. Rev. D 84 (2011)]
- Δ: [Sanchis-Alepuz, Eichmann, Villalba-Chavez, Alkofer, Phys. Rev. D 84 (2011)]

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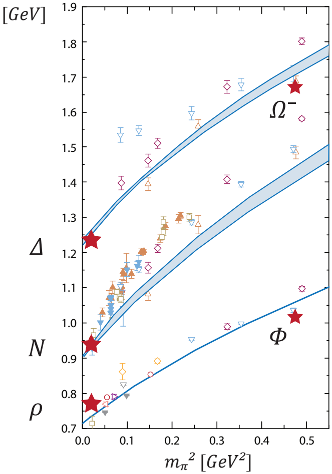


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- $\Delta$: [Sanchis-Alepuz, Eichmann, Villalba-Chavez, Alkofer, Phys. Rev. D 84 (2011)]
- $\rho$ meson: [Maris, Tandy, Phys. Rev. C 60 (1999)]
  $\rightsquigarrow$ Consistent description of baryons and mesons with one approximation.

# Quark-diquark approximation

3-body equation: transparent but numerically intricate (many Lorentz invariants and tensors)

Diquarks: [Barabanov et al., Prog.Part.Nucl.Phys. 116 (2021)]

- Physics: Diquark clustering in baryons? $\rightarrow$ Quark-diquark models in spirit of quark model
- Diquarks: From simple models to rich dynamical structure
- Quark-quark correlations in $T$ matrix

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## Derivation of 2-body equation

1. Neglect three-body interactions (approximation)
2. Replace scattering kernels $K$ by two-body matrices $T$ (exact)
3. Expansion in term of diquark correlations (approximation)

$\Rightarrow$ Fewer kinematic variables, smaller tensor basis (e.g., 8 instead of 64 for nucleon)

# Quark-diquark approximation

Faddeev equation:

- $\Gamma = \sum_i \Gamma_i = \sum_i K_i\, G_0\, \Gamma$
- Replace scattering kernels $K_i$ by two-body matrices $T_i$: $T_i = (1 + T_i\, G_0) K_i$
- $T_i\, G_0\, \Gamma = (1 + T_i\, G_0)\, \underbrace{K_i\, G_0\, \Gamma}_{\Gamma_i}$
- $\Gamma_i = T_i\, G_0(\Gamma - \Gamma_i) = T_i\, G_0(\Gamma_j + \Gamma_k)$

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Diquark approximation:

Quark-quark scattering matrix $\rightarrow$ sum over diquark correlations

Scalar and axialvector diquarks lightest $\rightarrow$ important in nucleon and $\Delta$

# Nucleon and $\Delta$

- Rainbow-ladder with Maris-Tandy interaction
- Parameters fixed in meson sector
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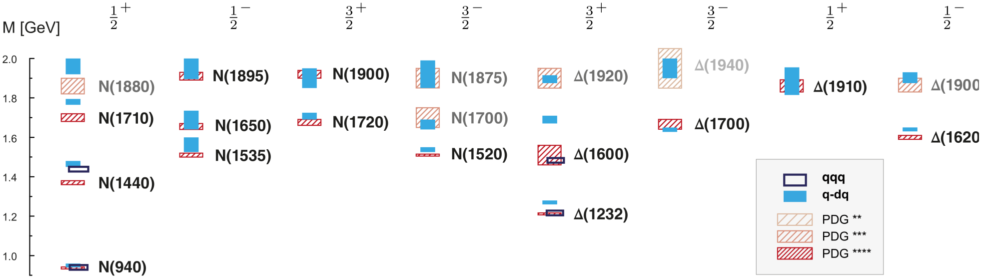
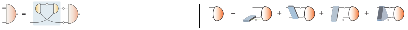



[Eichmann, Sanchis-Alepuz, Williams, Alkofer, Fischer, Prog.Part.Nucl.Phys. 91 (2016); Eichmann, Few Body Syst. 63 (2022)]

# Overview



## DSE/BSE/Faddeev landscape (2015)

level of complexity

| | | I) NJL/contact interaction | II) Quark-diquark model | III) DSE (RL) | | IV) DSE (bRL) |
|---|---|---|---|---|---|---|
| up/down, $P=\pm$ | $N,\Delta$ masses | ✓ | ✓ | ✓ | ✓ | ✓ |
| | $N,\Delta$ em. FFs | ✓ | ✓ | ✓ | ✓ | |
| | $N\to\Delta\gamma$ | ✓ | ✓ | ✓ | | |
| up/down, $P=+$ | $N^*,\Delta^*$masses | ✓ | ✓ | | | |
| | $\gamma N\to N^*/\Delta^*$ | ✓ | ✓ | | | |
| up/down, $P=-$ | $N^*,\Delta^*$masses | | ✓ | | | |
| | $\gamma N\to N^*/\Delta^*$ | | | | | |
| strange | ground states | | ✓ | | | |
| | excited states | | | | | |
| | em. FF | | | | | |
| | TFFs | | | | | |
| c/b | ground states | | | | | |
| | excited states | | | | | |
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Christian Fischer (University of Gießen)    Hadron physics with functional methods    71

[Fischer, Lecture at Internationale Universtitätswochen für Theoretische Physik, Admont, 2017]

# Overview

## DSE/BSE/Faddeev landscape (2021)

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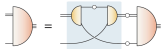

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| | | em. FF | | | ✓ | ✓ | |
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Christian Fischer (University of Gießen) Hadron physics with functional methods 72

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# Tetraquarks

- Experimental discovery of exotic XZY states $\rightarrow$ four-quark states?

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  History of $\sigma$ meson, lightest scalar nonet is incompatible with $q\,\overline{q}$ picture:

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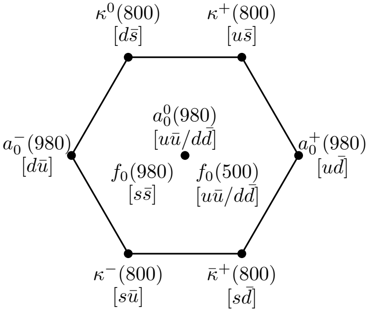

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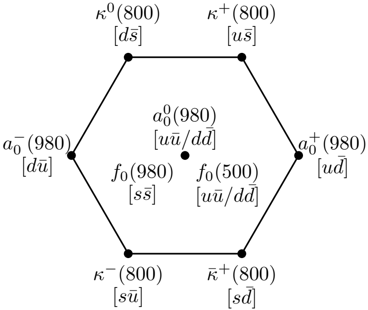
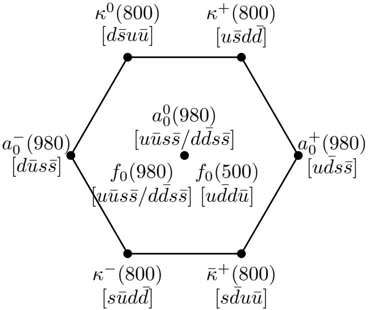



$q\,\overline{q}\,q\,\overline{q}$:



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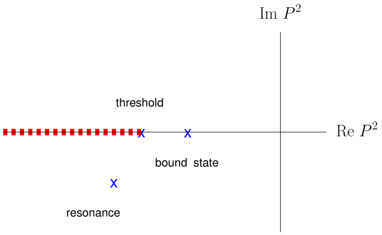
Tetraquark picture confirmed by functional calculations [Heupel, Eichmann, Fischer, Phys. Lett. B 718 (2012); Eichmann, Fischer, Heupel, Phys. Lett. B 753, 282 (2016)]: $\sigma$(500) is (dominantly) a four-quark state

Mixing of $q\bar{q}$ and $q\bar{q}q\bar{q}$ states:



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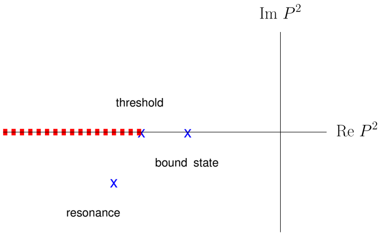
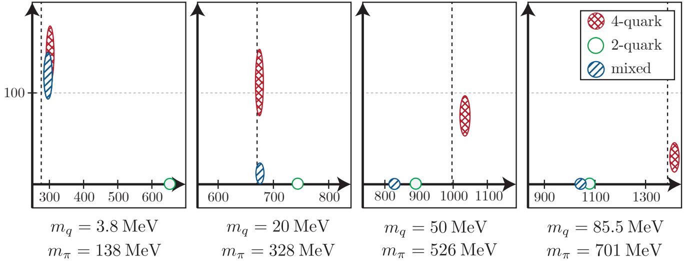


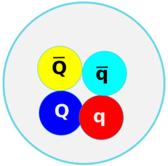




[Santowsky, Fischer, Phys. Rev. D 105 (2022)]

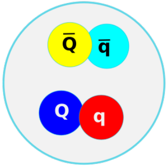
# Structure of four-quark states

Consider heavy-light system, e.g., $X(3872)$.

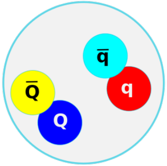
Possible clustering of states:



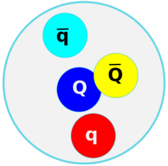
compact



diquark/antidiquark



meson molecule

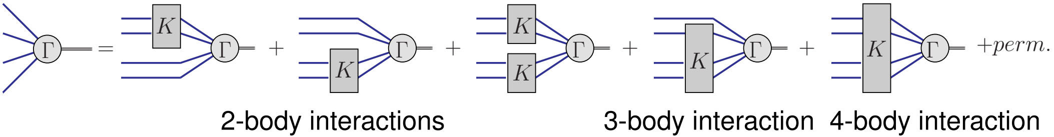


hadro-charmonium

Not mutually exclusive: Superpositions!

# Tetraquarks

[Eichmann, Fischer, Santowsky, Wallbott, Few-Body Syst.61 (2020)]



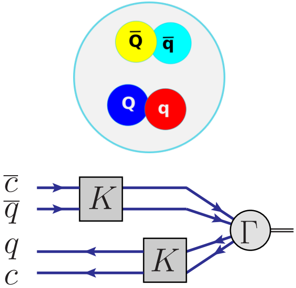

[Kvinikhidze, Khvedelidze, Theor. Math. Phys. 90 (1992); Heupel, Eichman, Fischer, PLB 718 (2012); Eichman, Fischer, Heupel, PLB 753 (2016)]

- Negelect 3- and 4-body interactions
- Complicated kinematics (4 momenta):
  - dressings $f$(9 Lorentz scalar)
  - scalar tetraquark: 256 tensors

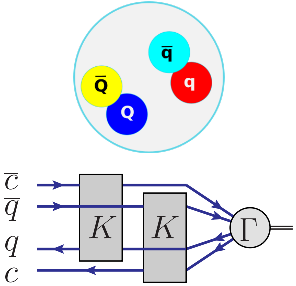
$\rightarrow$ Approximations necessary, e.g., only 2-body interactions

# Clustering

Dynamic distribution over different sectors:




diquark-antidiquark

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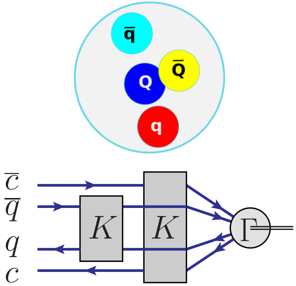


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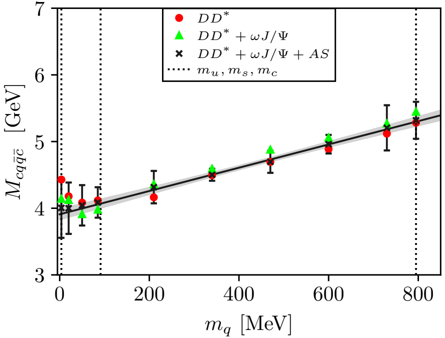
# $\chi_{c1}(3872)$ [$X(3872)$]




- Rainbow-ladder with Maris-Tandy
- Quark mass dependence
- $D\,D^*$: $c\bar{q}$, $q\bar{c}$ (molecule)
- $\omega\,J/\psi$: $c\bar{c}$, $q\bar{q}$ (hadrocharmonium)
- $AS$: $cq$, $\overline{cq}$ (diquark-antidiquark)

[Wallbott, Eichmann, Fischer, Phys. Rev. D 100 (2019)]

Heavy-light meson poles more important than diquark poles.

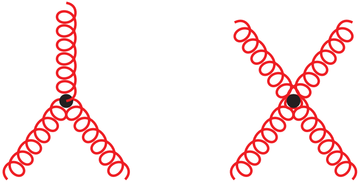
# Summary so far

- Up to now only rainbow-ladder with effective interaction (Maris-Tandy)
- Good quantitative description of pseudoscalar and vector mesons, nucleon and $\Delta$
- Insight into tetraquark composition
- Important: chiral symmetry $\rightarrow$ Goldstone bosons, mass creation. Encoded in axialvector WTI $\rightarrow$ nontrivial relations between quark selfenergy and kernels.

Beyond rainbow-ladder?

# Glueballs

What makes glueballs special?

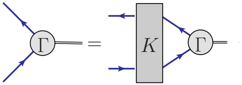


Mass dynamically created from massless (due to gauge invariance) gluons.

- No constituent matter particles $\rightarrow$ bound states of pure radiation
- Experimentally largely unexplored. Though a history of candidates. Recent results from $J/\psi$ decay: $f_0(1710)$, $f_0(1770)$ [Sarantsev, Denisenko, Thoma, Klempt, Phys. Lett. B 816 (2021); JPAC Coll., Rodas et al., Eur.Phys.J.C 82 (2022)]
- Theoretically not fully understood (existence, mixing, decays)

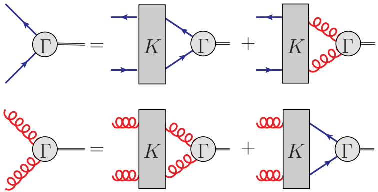
Experiment:
Production in glue-rich environments, e.g., $p\bar{p}$ annihilation (PANDA), pomeron exchange in $pp$ (central exclusive production), radiative $J/\psi$ decays

# Bound state equations for QCD




- Require scattering kernel $K$ and propagator.

# Bound state equations for QCD

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- Quantum numbers determine which amplitudes $\Gamma$ couple.

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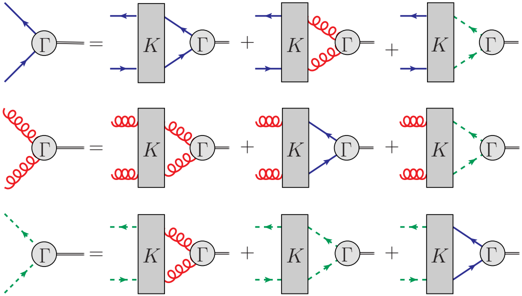


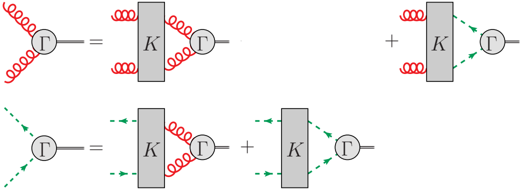

- Require scattering kernels $K$ and propagators.
- Quantum numbers determine which amplitudes $\Gamma$ couple.
- Ghosts from gauge fixing

**One framework**

- Natural description of mixing.
- Similar equations for hadrons with more than two constituents

# Bound state equations for QCD

Focus on pure glueballs.

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**One framework**

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- Similar equations for hadrons with more than two constituents

# 3PI effective action

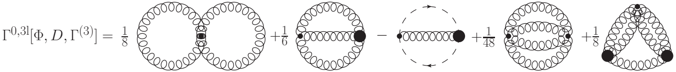
[Review: MQH, Phys.Rept. 879 (2020)]

Introduce sources for propagators and three-point functions into path integral and perform additional Legendre transformations:

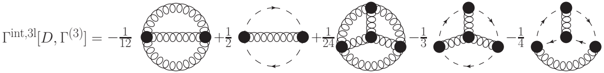
$$Z[J, R^{(2)}, R^{(3)}, \ldots] = \int D[\phi] e^{-S+\phi_i J_i + \frac{1}{2} R^{(2)}_{ij} \phi_i \phi_j + \frac{1}{3!} R^{(3)}_{ijk} \phi_i \phi_j \phi_k}$$

$$\Gamma[\Phi] \to \Gamma[\Phi, D, \Gamma^{(3)}]$$

3PI effective action truncated at three-loops: [Berges, PRD70 (2004); Carrington, Guo, PRD83 (2011)]




$$\Gamma^{3l}[\Phi, D, \Gamma^{(3)}] = \Gamma^{0,3l}[\Phi, D, \Gamma^{(3)}] + \Gamma^{int,3l}[\Phi, D, \Gamma^{(3)}]$$

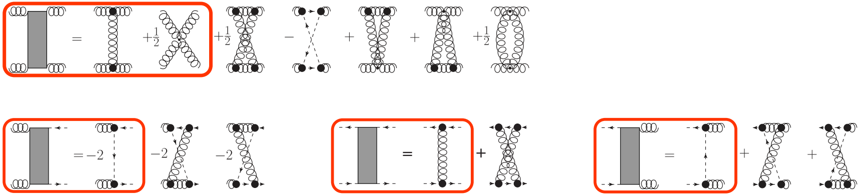


# Kernel construction

$$K = -2\frac{\delta^2 \Gamma^{3l}}{\delta D^2}$$

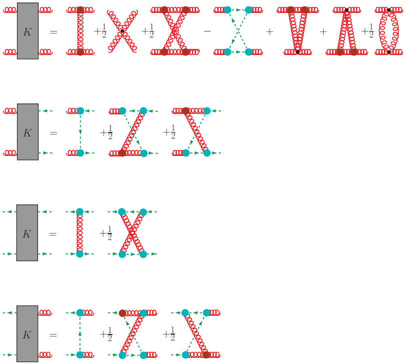
$\rightarrow$ Kernels constructed by cutting two legs: gluon/gluon,ghost/gluon, gluon/ghost, ghost/ghost

[Fukuda, Prog. Theor. Phys 78 (1987); McKay, Munczek, Phys. Rev. D 40 (1989); Sanchis-Alepuz, Williams, J. Phys: Conf. Ser. 631 (2015); MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C80 (2020)]



# Kernels

Systematic derivation from 3PI effective action: [Berges, PRD70 (2004); Carrington, Guo, PRD83 (2011)]
Self-consistent treatment of 3-point functions requires 3-loop expansion.

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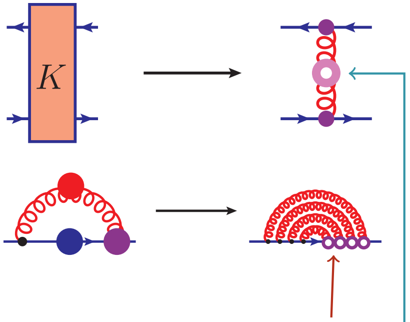
# Reminder: Functional spectrum calculations in rainbow-ladder truncation

Success in describing many aspects of the hadron spectrum qualitatively and quantitatively (mostly) based on rainbow-ladder truncation!

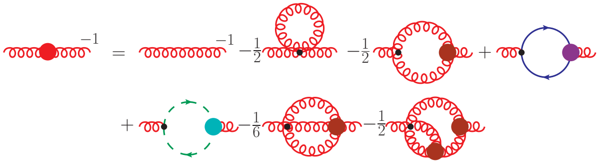
Workhorse for more than 20 years: Rainbow-ladder truncation with an effective interaction, e.g., Maris-Tandy (or similar).




restricted structure of equations ($\Gamma_\mu \to \gamma_\mu$)

# Functional glueball calculations

Glueballs? Rainbow-ladder?



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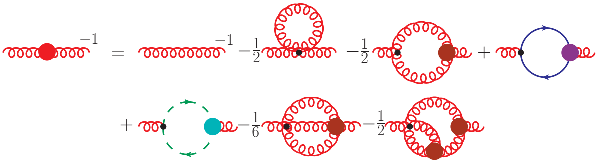

$\boxed{\text{There is no rainbow for gluons!}}$

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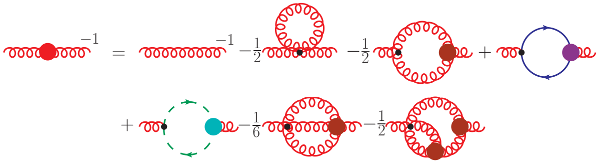


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Model based BSE calculations ($J = 0$):

- [Meyers, Swanson, Phys.Rev.D87 (2013)]
- [Sanchis-Alepuz, Fischer, Kellermann, von Smekal, Phys.Rev.D92, (2015)]
- [Souza et al., Eur.Phys.J.A56 (2020)]
- [Kaptari, Kämpfer, Few Body Syst.61 (2020)]

# Functional glueball calculations

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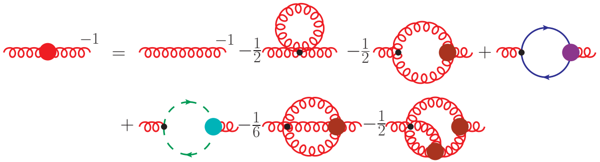


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Alternative: Calculated input [MQH, Phys.Rev.D 101 (2020)]

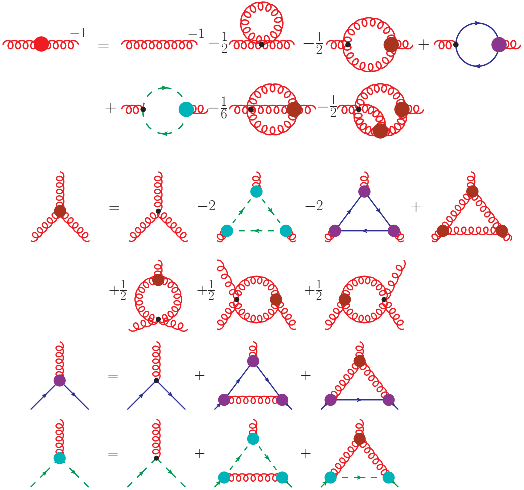
- $J = 0$: [MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C80 (2020)]
- $J = 0, 2, 3, 4$: [MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C81 (2021)]

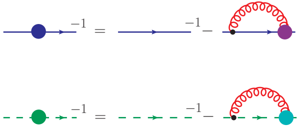
Extreme sensitivity on input!

# Correlation functions of quarks and gluons

## Equations of motion: 3-loop 3PI effective action

→ [Review: MQH, Phys.Rept. 879 (2020)]





- Conceptual and technical challenges: nonperturbative renormalization, two-loop diagrams, convergence, size of kernels, . . .
- Self-contained: Only parameters are the strong coupling and the quark masses!
- Long way, e.g., ghost-gluon vertex, three-gluon vertex, four-gluon vertex, . . . ,
- → MQH, Phys.Rev.D 101 (2020)

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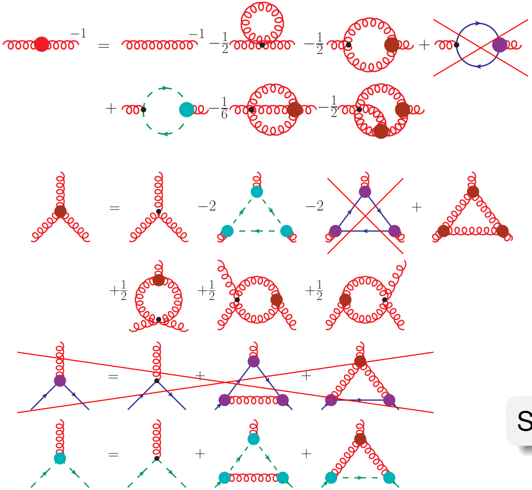

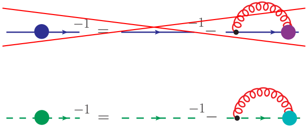

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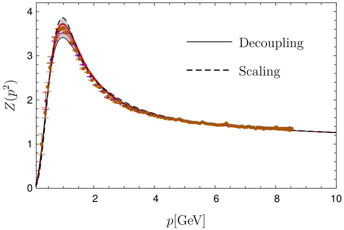
- Long way, e.g., ghost-gluon vertex, three-gluon vertex, four-gluon vertex, . . . ,
- → MQH, Phys.Rev.D 101 (2020)

Start with pure gauge theory.

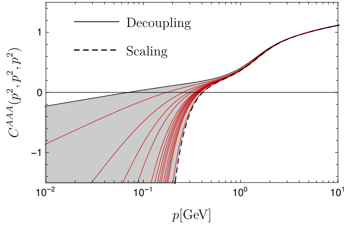
# Landau gauge propagators

Self-contained: Only external input is the coupling! [MQH, Phys.Rev.D 101 (2020)]

Gluon dressing function:



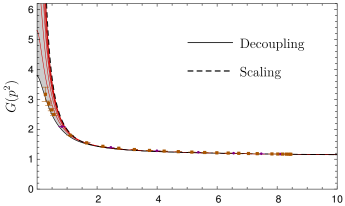

Three-gluon vertex:




Family of solutions [von Smekal, Alkofer, Hauck, PRL79 (1997); Aguilar, Binosi, Papavassiliou, Phys.Rev.D 78 (2008); Boucaud et al., JHEP06 (2008); Fischer, Maas, Pawlowski, Ann.Phys. 324 (2008); Alkofer, MQH, Schwenzer, Phys. Rev. D 81 (2010)]

Nonperturbative completions of Landau gauge [Maas, Phys. Lett. B 689 (2010)]?

Ghost dressing function:




Ghost-gluon vertex

Three-gluon vertex

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- Agreement with lattice results. ✓

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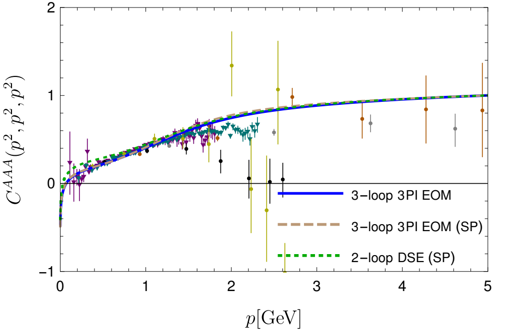
3PI vs. 2-loop DSE:



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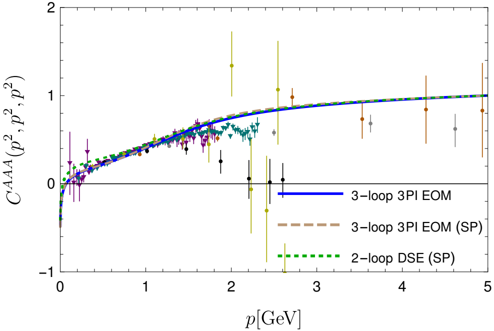
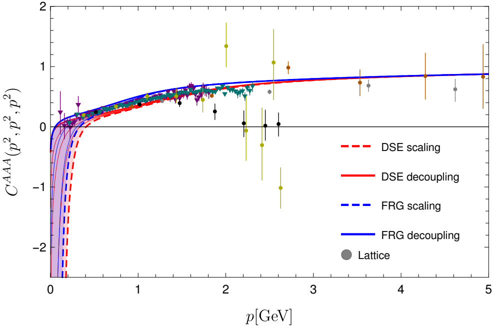



DSE vs. FRG:




[Cucchieri, Maas, Mendes, Phys.Rev.D77 (2008); Sternbeck et al., Proc.Sci. LATTICE2016 (2017); Cyrol et al., Phys.Rev.D 94 (2016); MQH, Phys.Ref.D101 (2020)]

# Stability of the solution: Extensions

- Three-gluon vertex: Tree-level dressing dominant, others subleading [Eichmann, Williams, Alkofer, Vujinovic, Phys.Rev.D89 (2014); Pinto-Gómez et al., 2208.01020] ✓

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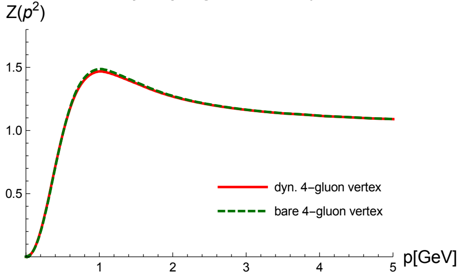
- Four-gluon vertex: Influence on propagators tiny for $d = 3$ [MQH, Phys.Rev.D93 (2016)]



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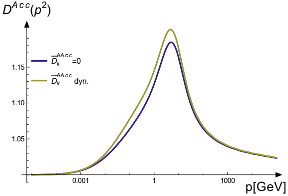
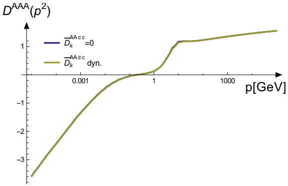
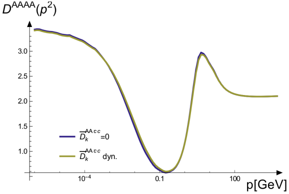
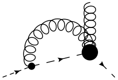
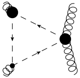
- Two-ghost-two-gluon vertex [MQH, Eur. Phys.J.C77 (2017)]: ✓
  (FRG: [Corell, SciPost Phys. 5 (2018)])



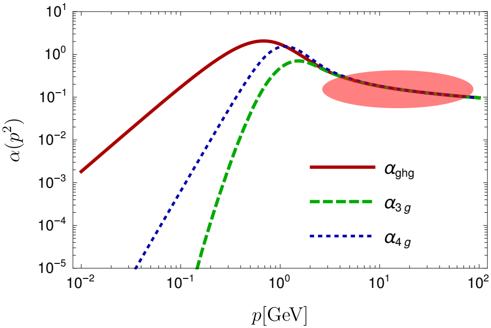


# Gauge invariance

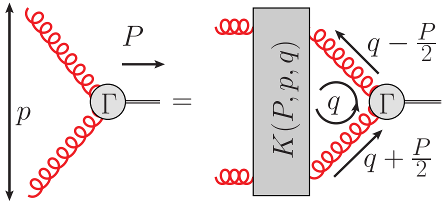
[MQH, Phys. Rev. D 101 (2020)]

Couplings can be extracted from each vertex.

- Slavnov-Taylor identities (gauge invariance): Agreement perturbatively (UV) necessary.
  [Cyrol et al., Phys.Rev.D 94 (2016)]
- Difficult to realize: Small deviations $\rightarrow$ Couplings cross and do not agree.
- Here: Vertex couplings agree down to GeV regime (IR can be different).



# Correlation functions for complex momenta




(pseudoscalar glueball)

$$\boldsymbol{\lambda(P)}\Gamma(P) = \mathcal{K}\cdot\Gamma(P)$$

$\rightarrow$ Eigenvalue problem for $\Gamma(P)$:

1. Solve for $\lambda(P)$.
2. Find $P$ with $\lambda(P) = 1$.
   $\Rightarrow M^2 = -P^2$

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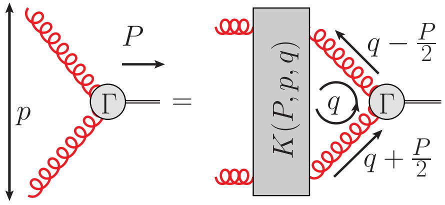


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However:

Propagators are probed at $\left(q\pm\frac{P}{2}\right)^2 = \frac{P^2}{4} + q^2 \pm \sqrt{P^2\,q^2}\cos\theta = -\frac{M^2}{4} + q^2 \pm \boldsymbol{i}\,M\sqrt{q^2}\cos\theta$

$\rightarrow$ Complex for $P^2<0$!

Time-like quantities ($P^2<0$) $\rightarrow$ Correlation functions for complex arguments.

# Correlation functions in the complex plane

Propagator results

Standard integration techniques fail. ↯

$$\int d^4q \to \int_{\Lambda_{IR}^2}^{\Lambda_{UV}^2} dq^2 \int d\theta_1$$

Consider example integral:

$$K(p^2) = \int dq^2 J(q^2, p^2), \quad J(p^2, q^2) = \int dq^2 \int d\theta \sin^2\theta_1 \frac{1}{q^2 + p^2 + \sqrt{p^2}\sqrt{q^2}\cos\theta_1 + m^2} \frac{1}{q^2 + m^2}$$

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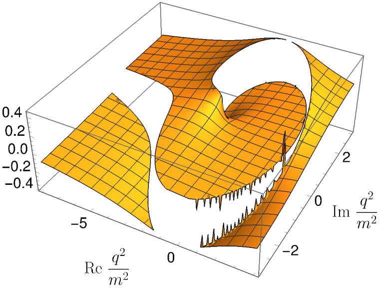
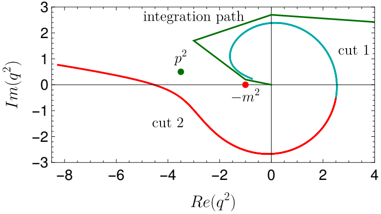
After $\theta_1$ integration:




Integration path $\Lambda_{IR}^2 \to \Lambda_{UV}^2$ on real line forbidden.



# Extrapolation of $\lambda(P^2)$

## Extrapolation method

- Extrapolation to time-like $P^2$ using Schlessinger's continued fraction method (proven superior to default Padé approximants) [Schlessinger, Phys.Rev.167 (1968)]
- Average over extrapolations using subsets of points for error estimate

$$f(x) = \frac{f(x_1)}{1 + \frac{a_1(x-x_1)}{1+\frac{a_2(x-x_2)}{1+\frac{a_3(x-x_3)}{\cdots}}}}$$

Coefficients $a_i$ can determined such that $f(x)$ exact at $x_i$.

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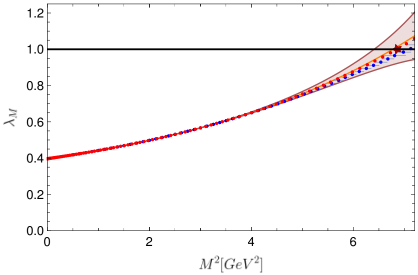
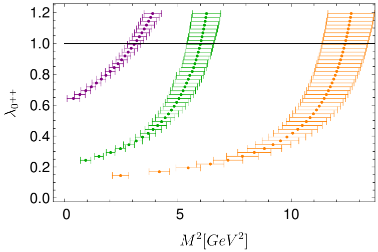
Test extrapolation for solvable system:

Heavy meson [MQH, Sanchis-Alepuz, Fischer, Eur.Phys.J.C 80 (2020)]

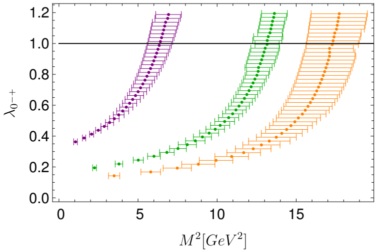


# Extrapolation for glueball eigenvalue curves

$0^{++}$:



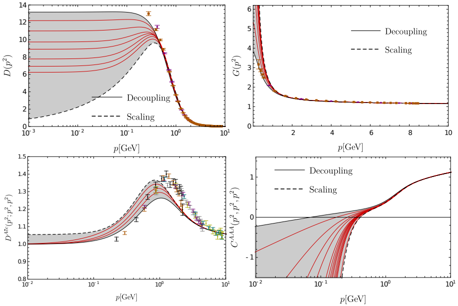

$0^{-+}$:




Several curves: ground state and excited states.

# Glueball results J=0

Gauge-variant correlation functions:



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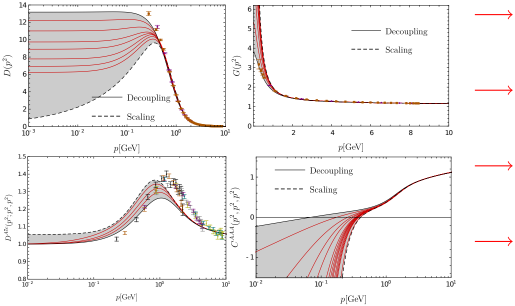
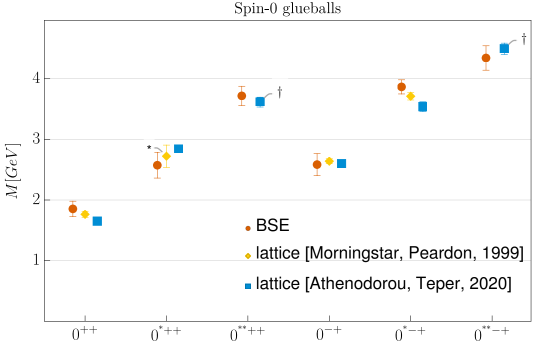



## Unique physical spectrum:



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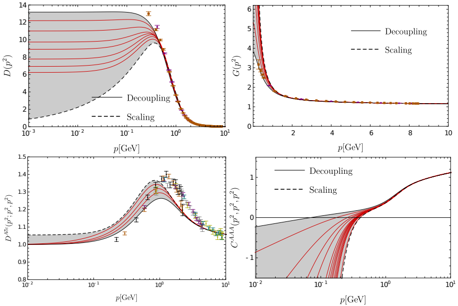


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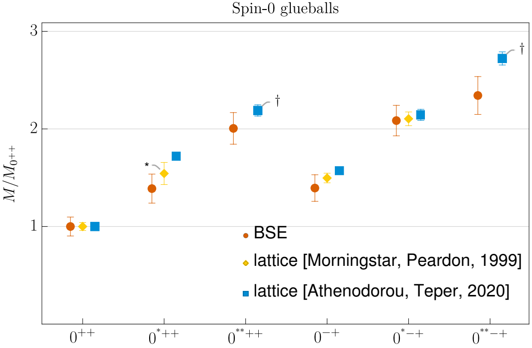
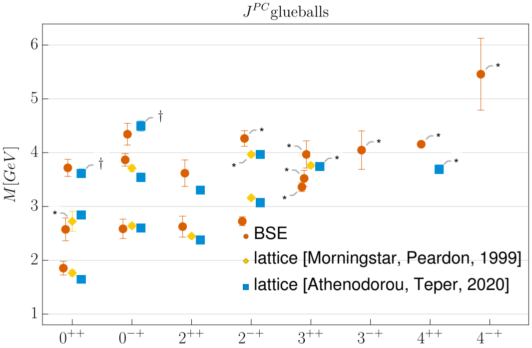


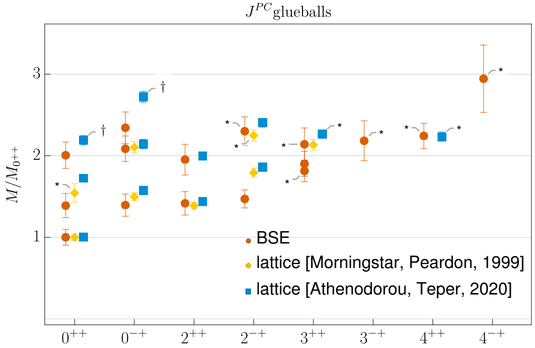

Spectrum independent! $\rightarrow$ Family of solutions yields the same physics.

All results for $r_0 = 1/418(5)$ MeV.

[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C80 (2020)]

# Glueball results



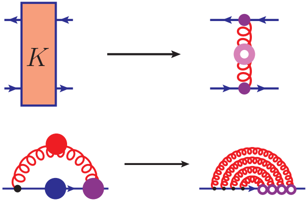

[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C81 (2021)]

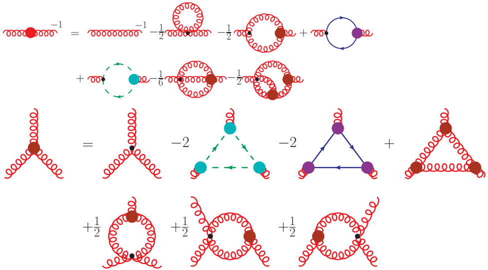
*: identification with some uncertainty
†: conjecture based on irred. rep of octahedral group

- Agreement with lattice results
- (New states: $0^{**++}$, $0^{**-+}$, $3^{-+}$, $4^{-+}$)

# Model vs. first-principle calculations

| Bottom-up | Top-down |
|---|---|
| Models | Direct calculations |
| Parameters: dependent (-), tuning (+) | No parameters: independent (+), no tuning (-) |
| Often simpler | Typically more involved |
| Well-tried and successful for certain applications | Requires good control and tests of input |
| Good to test qualitative understanding | Results form first principles possible |

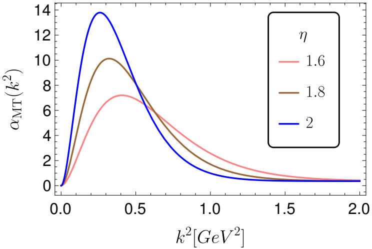


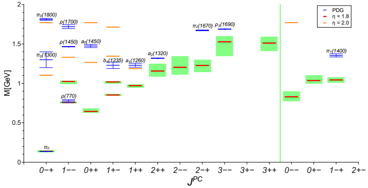



(Mixtures)

# Summary

- Model-based calculations:
  - Meson and baryon spectrum
  - Tetraquarks: Scalar multiplet, clustering



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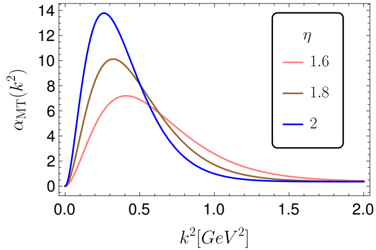
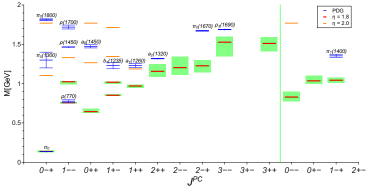
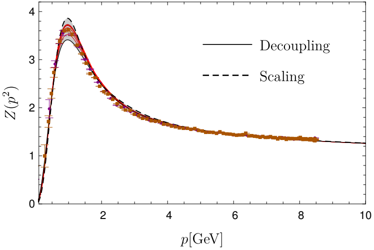
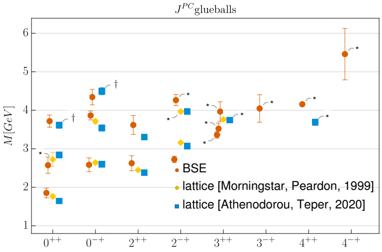



- From first principles:
  - Input: agreement with other methods (lattice + continuum) and extensions tested
  - Quantitative results for glueball spectrum




Thank you for your attention.

# Derivation of DSEs (details) I

Back

Integral of a total derivative vanishes:

$$0 = \int D[\phi] \frac{\delta}{\delta\phi} e^{-S+\int dy \phi(y) J(y)} = \int D[\phi] \left( -\frac{\delta S}{\delta\phi(x)} + J(x) \right) e^{-S+\int dy \phi(y) J(y)}$$

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Pull in front of integral $\to$ Master DSE for full correlation functions

$$0 = \left( -\frac{\delta S}{\delta \phi(x)} \Bigg|_{\phi(x')=\delta/\delta J(x')} + J(x) \right) \underbrace{Z[J]}_{e^{W[J]}} = 0$$

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$$0 = \int D[\phi] \frac{\delta}{\delta\phi} e^{-S+\int dy \phi(y) J(y)} = \int D[\phi] \left(-\frac{\delta S}{\delta\phi(x)} + J(x)\right) e^{-S+\int dy \phi(y) J(y)}$$

Pull in front of integral $\rightarrow$ Master DSE for full correlation functions

$$0 = \left(-\frac{\delta S}{\delta\phi(x)}\bigg|_{\phi(x')=\delta/\delta J(x')} + J(x)\right) \underbrace{Z[J]}_{e^{W[J]}} = 0$$

$$e^{-W[J]}\left(\frac{\delta}{\delta J(x)}\right) e^{W[J]} = \frac{\delta W[J]}{\delta J(x)} + \frac{\delta}{\delta J(x)}$$

$\rightarrow$ Master DSE for connected correlation functions

$$-\frac{\delta S}{\delta\phi(x)}\bigg|_{\phi(x')=\frac{\delta W[J]}{\delta J(x')}+\frac{\delta}{\delta J(x')}} + J(x) = 0.$$

# Derivation of DSEs (details) II

Legendre transformation:

$$\frac{\delta W[J]}{\delta J(x)} \to \Phi(x)$$

$$\frac{\delta}{\delta J(x)} \to \int dz\, D(x,z)^J \frac{\delta}{\delta \Phi(z)}$$

$$\left( \frac{\delta}{\delta J(x)} = \int dz \frac{\delta \Phi(z)}{\delta J(x)} \frac{\delta}{\delta \Phi(z)} = \int dz \frac{\delta}{\delta J(x)} \frac{\delta W[J]}{\delta J(z)} \frac{\delta}{\delta \Phi(z)} = \int dz \frac{\delta^2 W[J]}{\delta J(x) \delta J(z)} \frac{\delta}{\delta \Phi(z)} \right)$$

# Derivation of DSEs (details) II

Legendre transformation:

$$\frac{\delta W[J]}{\delta J(x)} \to \Phi(x)$$

$$\frac{\delta}{\delta J(x)} \to \int dz\, D(x,z)^J \frac{\delta}{\delta \Phi(z)}$$

$$\left(\frac{\delta}{\delta J(x)} = \int dz \frac{\delta \Phi(z)}{\delta J(x)} \frac{\delta}{\delta \Phi(z)} = \int dz \frac{\delta}{\delta J(x)} \frac{\delta W[J]}{\delta J(z)} \frac{\delta}{\delta \Phi(z)} = \int dz \frac{\delta^2 W[J]}{\delta J(x) \delta J(z)} \frac{\delta}{\delta \Phi(z)}\right)$$

## Master DSE for 1PI correlation functions

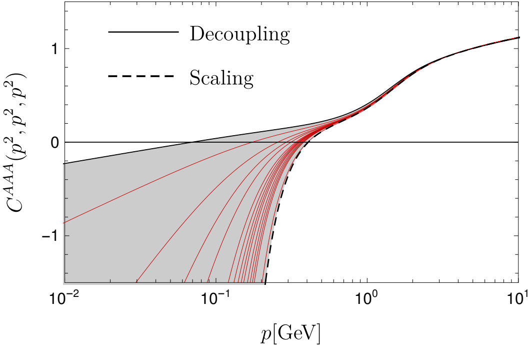
$$-\left.\frac{\delta S}{\delta \phi(x)}\right|_{\phi(x')=\Phi(x')+\int dz\, D(x',z)^J\, \delta/\delta\Phi(z)} + \frac{\delta \Gamma[\Phi]}{\delta \Phi(x)} = 0$$

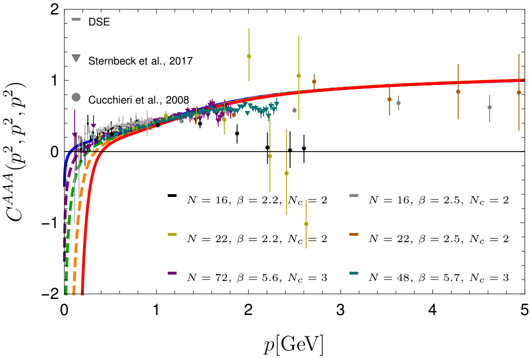
Get DSE for $n$-point function by applying $n-1$ derivatives.

# Three-gluon vertex

Back

[Cucchieri, Maas, Mendes, Phys. Rev. D 77 (2008); Sternbeck et al., 1702.00612; MQH, Phys. Rev. D 101 (2020)]



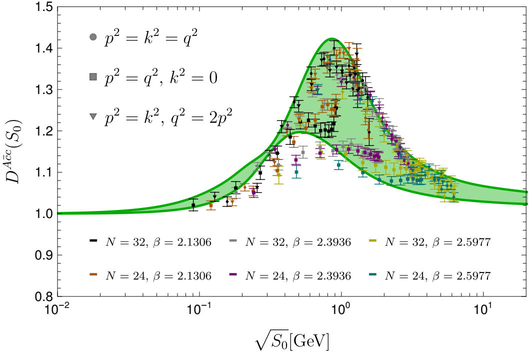




- Simple kinematic dependence of three-gluon vertex (only singlet variable of $S_3$)
- Large cancellations between diagrams

# Ghost-gluon vertex

Back

Ghost-gluon vertex:



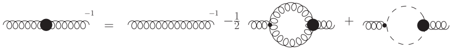

[Maas, SciPost Phys. 8 (2019);
MQH, Phys. Rev. D 101 (2020)]

- Nontrivial kinematic dependence of ghost-gluon vertex
- Qualitative agreement with lattice results, though some quantitative differences (position of peak!).

# Landau gauge propagators in the complex plane

Back

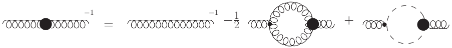
Simpler truncation:

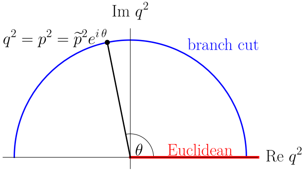


# Landau gauge propagators in the complex plane

Back

Simpler truncation:



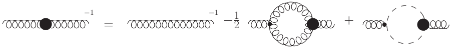


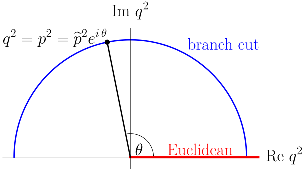

$\rightarrow$ Opening at $q^2 = p^2$.

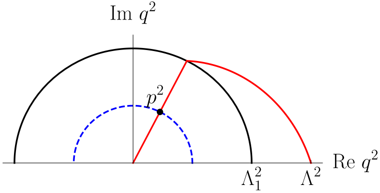
# Landau gauge propagators in the complex plane

Back

Simpler truncation:





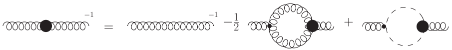




$\rightarrow$ Opening at $q^2 = p^2$.

Appearance of branch cuts for complex momenta forbids integration directly to cutoff.

# Landau gauge propagators in the complex plane

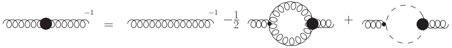
Simpler truncation:



[Fischer, MQH, Phys.Rev.D 102 (2020)]

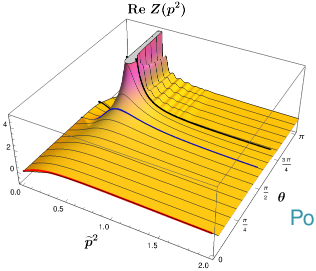
# Landau gauge propagators in the complex plane

Simpler truncation:

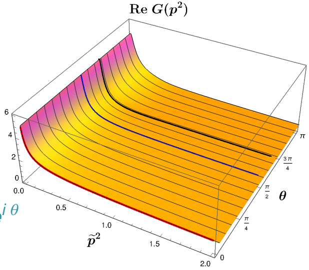



[Fischer, MQH, Phys.Rev.D 102 (2020)]

Ray technique for self-consistent solution of a DSE:

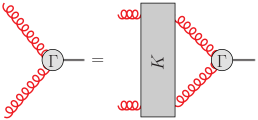


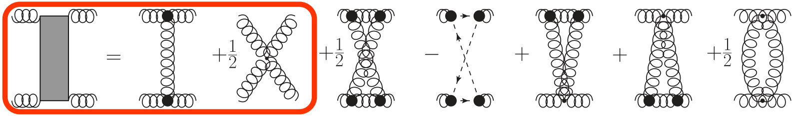

Polar coordinates: $p^2 = \tilde{p}^2 e^{i\,\theta}$




- Current truncation leads to a pole-like structure in the gluon propagator.
- Analyticity up to 'pole' confirmed by various tests (Cauchy-Riemann, Schlessinger, reconstruction)

# Higher order diagrams




One-loop diagrams only:

[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C80 (2020); MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C81 (2021)]

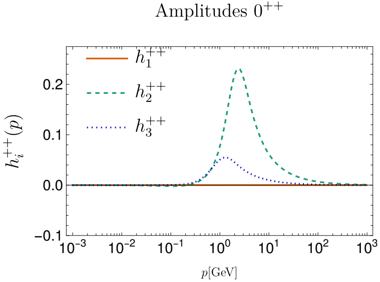
Two-loop diagrams: subleading effects

- $0^{-+}$: none
  [MQH, Fischer, Sanchis-Alepuz, EPJ Web Conf. 258 (2022)]
- $0^{++}$: $< 2\%$
  [MQH, Fischer, Sanchis-Alepuz, HADRON2021, arXiv:2201.05163]

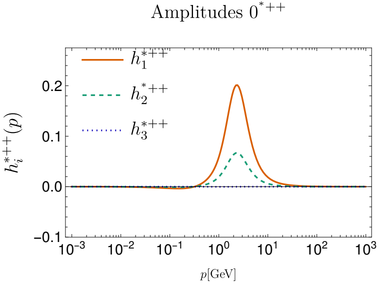
# Amplitudes

Eigenvectors of eigenvalue problem: Amplitudes, information about significance of single parts.

Ground state scalar glueball:




Excited scalar glueball:



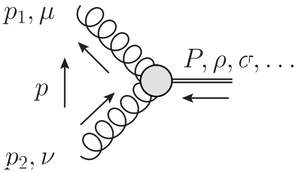

$\rightarrow$ Amplitudes have different behavior for ground state and excited state. Useful guide for future developments.

$\rightarrow$ Meson/glueball amplitudes: Information about mixing.

# Glueball amplitudes for spin $J$

[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C81 (2021)]

$$\Gamma_{\mu\nu\rho\sigma\ldots}(p_1,p_2) = \sum \tau^i_{\mu\nu\rho\sigma\ldots}(p_1,p_2)h_i(p_1,p_2)$$




Increase in complexity:

- 2 gluon indices (transverse)
- $J$ spin indices (symmetric, traceless, transverse to $P$)

Numbers of tensors:

| $J$ | $\mathrm{P}=+$ | $\mathrm{P}=-$ |
|---|---|---|
| 0 | 2 | 1 |
| 1 | 4 | 3 |
| >2 | 5 | 4 |

Low number of tensors, but high-dimensional tensors!

$\rightarrow$ Computational cost increases with $J$.

# $J = 1$ glueballs

**Landau-Yang theorem**

Two-photon states cannot couple to $J^P = \mathbf{1}^{\pm}$ or $(2n+1)^-$
[Landau, Dokl.Akad.Nauk SSSR 60 (1948); Yang, Phys. Rev. 77 (1950)].
($\rightarrow$ Exclusion of $J = 1$ for Higgs because of $h \rightarrow \gamma\gamma$.)

Applicable to glueballs?

$\rightarrow$ Not in this framework, since gluons are not on-shell.

$\rightarrow$ Presence of $J = 1$ states is a dynamical question.

$J = 1$ not found here.

# Charge parity

Transformation of gluon field under charge conjugation:

$$A^a_\mu \to -\eta(a) A^a_\mu$$

where

$$\eta(a) = \begin{cases} +1 & a = 1,3,4,6,8 \\ -1 & a = 2,5,7 \end{cases}$$

Color neutral operator with two gluon fields:

$$A^a_\mu A^a_\nu \to \eta(a)^2 A^a_\mu A^a_\nu = A^a_\mu A^a_\nu.$$

$\Rightarrow C = +1$

Negative charge parity, e.g.:

$$\begin{aligned} d^{abc} A^a_\mu A^b_\nu A^c_\rho \to &- d^{abc} \eta(a)\eta(b)\eta(c) A^a_\mu A^b_\nu A^c_\rho = \\ &- d^{abc} A^a_\mu A^b_\nu A^c_\rho. \end{aligned}$$

Only nonvanishing elements of the symmetric structure constant $d^{abc}$: zero or two indices equal to 2, 5 or 7.