Using Dyson-Schwinger equations to investigate Yang-Mills theory in the infrared: scaling solutions, power counting and infrared exponents

Markus Q. Huber, Reinhard Alkofer, Kai Schwenzer

Institute for Physics, Karl-Franzens-University Graz

Oct. 1, 2009

Seminar, Instituto de Física, Universidade Federal Fluminense

Der Wissenschaftsfonds.

SIC!QFT

Table of contents

Phenomenology of the strong interaction

Particles under the influence of the strong force: hadrons, e. g. π, K, η, proton, neutron, Λ, Σ, ...

- High energy experiments: point-like particles inside the hadrons (quarks).
- Quarks only exist in bound states, never as free particles (confinement).
- Mediator of the strong force: gluons (also confined).
- Theory: Quantum Chromodynamics (QCD).
- At high energies QCD is asymptotically free, i. e. the coupling gets small and we can "observe" quarks (Nobel prize 2004).
- At lower energies non-perturbative methods are needed.

This talk

In this talk I will focus on the low energy behavior of Yang-Mills theory (gluonic part of QCD).

Confinement of quarks and gluons

- Confinement is a long-range $\leftrightarrow$ IR phenomenon: We do not see individual $\sim$ infinitely separated quarks or gluons.
- One expects that the property of being confined is encoded in the particles' propagators.
- Different confinement criteria for the propagators:
  - Positivity violations: negative norm contributions $\rightarrow$ not a particle of the physical state space
  - Gribov-Zwanziger (Landau gauge, Coulomb gauge): IR suppression of the gluon propagator $\rightarrow$ no long-distance propagation
  - Kugo-Ojima: quartet mechanism, e. g. Gupta-Bleuler formalism in QED: timelike and longitudinal photon cancel each other

Functional methods employ

correlation functions/Green fcts./n-point fcts./propagators and vertices.

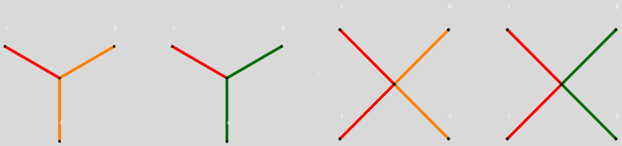
The equations of motion of these are the Dyson-Schwinger equations.

Propagators and vertices

The theory is encoded in the Green functions: "building blocks" for functional equations.
They describe propagation and interactions of fields.

Graphical notation

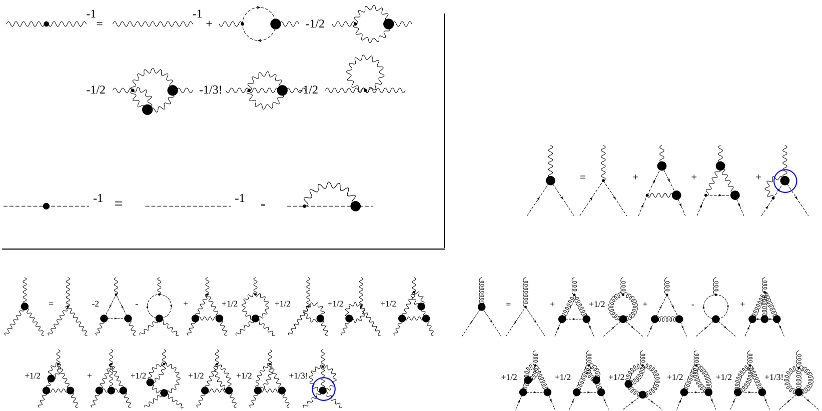
Propagators: 

Vertices:  , ...

The propagators and interactions are given by the Lagrangian of the theory.
Shorthand notation: propagator of field A is AA, quartic interaction is $AAAA$ etc.

The tower of DSEs

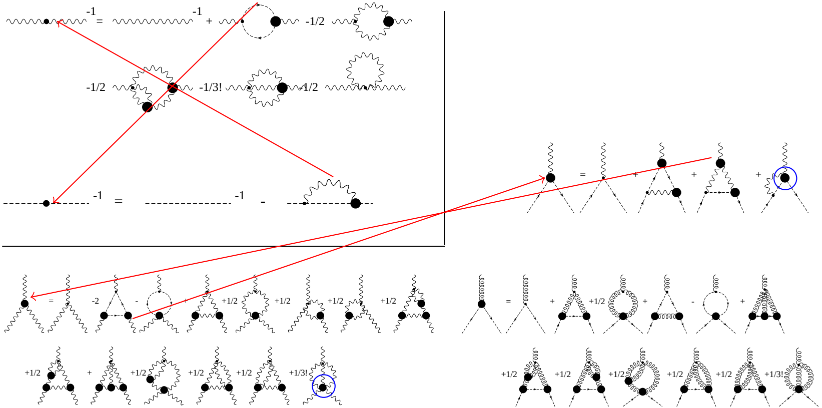
DSE describe non-perturbatively how particles propagate and interact.



n-point functions couple to n-point, (n+1)- and (n+2)-point functions
⇒ truncations?

The tower of DSEs

DSE describe non-perturbatively how particles propagate and interact.

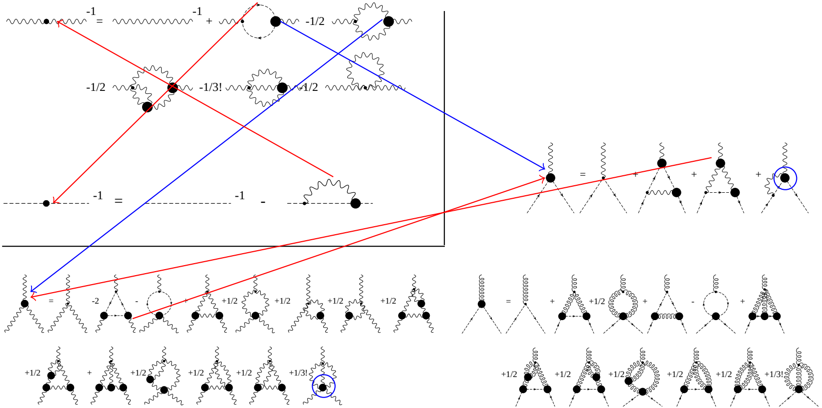


n-point functions couple to n-point, (n+1)- and (n+2)-point functions

$\Rightarrow$ truncations?

The tower of DSEs

DSE describe non-perturbatively how particles propagate and interact.



n-point functions couple to n-point, (n+1)- and (n+2)-point functions
⇒ truncations?

Deriving Dyson-Schwinger equations

Starting from the translation invariance of the path integral,

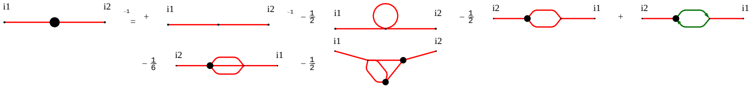
$$\frac{\delta}{\delta\phi}Z[J]=\int[D\phi]\left(J-\frac{\delta S}{\delta\phi}\right)e^{-S+J\,\Phi}=0,$$

the DSEs for all Green functions (full, connected, 1PI) can be derived by further differentiations.
After the first derivative is done, the procedure is iterative. Doing it by hand becomes tedious.

For example: Landau gauge, only 2 propagators, 3 interactions

Landau Gauge: Propagators

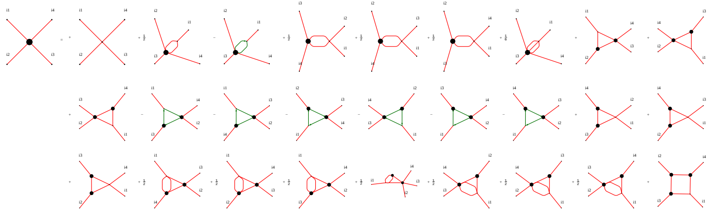
Gluon propagator:

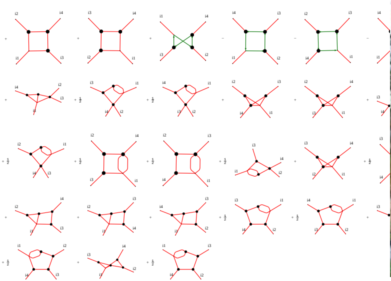


Ghost propagator:

Landau Gauge: Four-Gluon Vertex

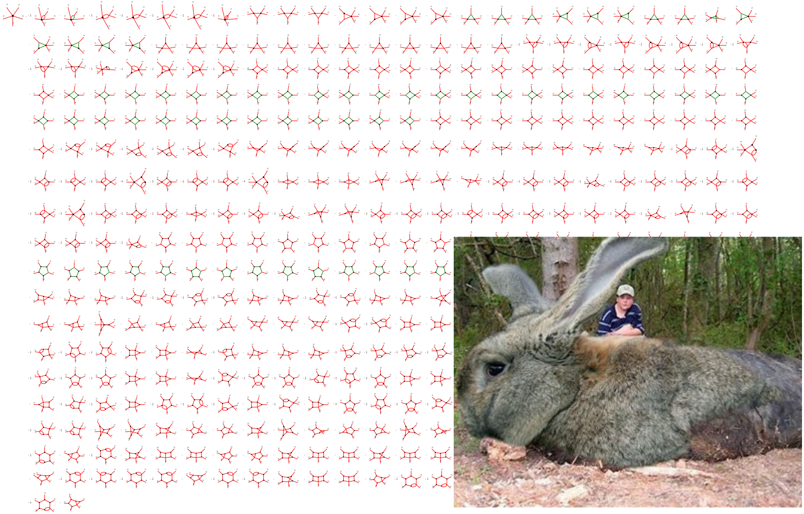
66 terms





Landau Gauge: Five-Gluon Vertex

434 terms



DoDSE

⇒ *DoDSE* [Alkofer, M.Q.H., Schwenzer, CPC 180 (2009)]

Given a structure of interactions, the DSEs are derived symbolically using *Mathematica*.

Example (Landau gauge):

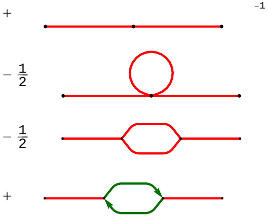
- only input: interactions in Lagrangian (AA, AAA, AAAA, cc, Acc)
- Which DSE do I want?

- Step-by-step calculations possible.
- Can handle mixed propagators (then there are really many diagrams).

Upgrade: *Symb2Alg* produces algebraic from the symbolic expressions.

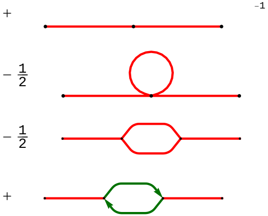
DoDSE upgrade: *Symb2Alg*

Symbolic expressions are fine for some tasks, e. g. infrared analysis, but also algebraic expressions are needed!



DoDSE upgrade: *Symb2Alg*

Symbolic expressions are fine for some tasks, e. g. infrared analysis, but also algebraic expressions are needed!



$$\frac{\left(g^{\mu 1\,\nu 1}\,\mathrm{p1}^2 - \mathrm{p1}^{\mu 1}\,\mathrm{p1}^{\nu 1}\right)\,\delta_{\mathrm{a1\,b1}}}{{\mathrm{p1}^2}^2} - \frac{C_A\,g^2\,\left(\mathrm{q1}^{\mu 1}\,\mathrm{q1}^{\nu 1} + 2\,g^{\mu 1\,\nu 1}\,\mathrm{q1}^2\right)\,\delta_{\mathrm{a1\,b1}}}{{\mathrm{q1}^2}^2} - \frac{C_A\,g^2\,\left(\ll 1 \gg\right)\,\delta_{\mathrm{a1\,b1}}}{2\,{\ll 2 \gg^2}^2\,\left(\mathrm{p1}^2 + 2\,\mathrm{p1}\cdot \ll 2 \gg + \mathrm{q1}^2\right)^2} + \frac{C_A\,g^2\,\mathrm{q1}^{\mu 1}\,\left(\mathrm{p1}^{\nu 1} + \mathrm{q1}^{\nu 1}\right)\,\delta_{\mathrm{a1\,b1}}}{\mathrm{q1}^2\,\left(\mathrm{p1}^2 + 2\,\mathrm{p1}\cdot\mathrm{q1} + \mathrm{q1}^2\right)}$$

Mathematica package *Symb2Alg*: Transforms output of *DoDSE* into algebraic expressions.
Depending on Feynman rules compatible with *FeynCalc*.

Dyson-Schwinger equations (DSEs) for investigating QCD

Infinitely large tower of equations

Equations of motion of Green functions

Dyson-Schwinger equations (DSEs) for investigating QCD

Infinitely large tower of equations

Equations of motion of Green functions

Pros:

- Exact equations
  $\rightarrow$ non-perturbative regime accessible
- Continuum, different scales accessible
  $\rightarrow$ complement lattice method

Dyson-Schwinger equations (DSEs) for investigating QCD

Infinitely large tower of equations

Equations of motion of Green functions

Pros:

- Exact equations
  $\rightarrow$ non-perturbative regime accessible
- Continuum, different scales accessible
  $\rightarrow$ complement lattice method

Cons?:

- Truncations
- Gauge-dependent

Dyson-Schwinger equations (DSEs) for investigating QCD

Infinitely large tower of equations

Equations of motion of Green functions

Pros:

- Exact equations
  $\rightarrow$ non-perturbative regime accessible
- Continuum, different scales accessible
  $\rightarrow$ complement lattice method

Cons?:

- Truncations (not for all tasks)
- Gauge-dependent
  $\rightarrow$ Exploit advantages of different gauges

About infrared propagators in Landau gauge

Yang-Mills theory's Green functions are best investigated in Landau gauge:

- Decoupling solution: IR constant gluon propagator, tree-level ghost propagator
- Scaling solution: IR dressing functions characterized by power laws, exponents related by scaling relation, ghost IR enhanced, gluon IR vanishing

Different methods

- Most lattice calculations find the decoupling scenario.
- Functional equations use boundary conditions to get either solution.
- The Gribov-Zwanziger action yields the scaling solution, which can be altered to the decoupling type by the addition of condensates (refined GZ framework).

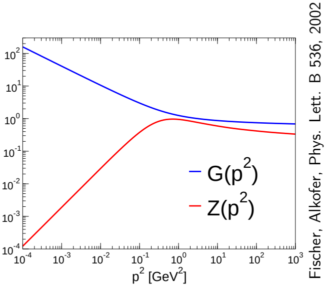
Knowledge about the IR behavior: useful for numerical calculations, test some confinement scenarios.

Infrared propagators in Landau gauge: Decoupling solution

- Lattice produces decoupling type solution [e. g. Cucchieri, Mendes, PoS(LAT) 2007; Bogolubsky, Ilgenfritz, Müller-Preussker PoS(LAT) 2007]
- exceptions: $d = 2$ [Maas, PRD75], $\beta = 0$ [Sternbeck, von Smekal, PoS(LAT) 2008]
- refined Gribov-Zwanziger framework [Dudal, Sorella, Vandersickel, Verschelde, PRD77]: introduce condensates
- first results using partly Dyson-Schwinger equations [Boucaud et al., JHEP06(2008)], used input from lattice for gluon propagator
- obtained by modified DSEs [Aguilar, Binosi, Papavassiliou, PRD 78]
- full solution of propagators DSEs/RGEs [Fischer, Maas, Pawlowski, 0810.1987]
- higher vertex functions are not IR enhanced [Alkofer, M.Q.H., Schwenzer, 0801.2763]

Always a family of solutions is found.

Infrared propagators in Landau gauge: Scaling solution



- Qualitative solution for the whole tower of vertex functions known [Alkofer, Fischer, Llanes-Estrada, PLB 611 (2005); M.Q.H., Alkofer, Fischer, Schwenzer,PLB 659 (2008)].
- Details about three- and four-point functions known, [e. g. Kellermann, Fischer, PRD 78 (2008); Alkofer, M.Q.H., Schwenzer, EPJC 62 (2009), 0801.2762].
- In agreement with Kugo-Ojima and Gribov-Zwanziger scenarios.
- Positivity violating propagators.

Maas, 0907.5185: Alternative algorithm to choose gauge copy can produce IR enhancement of the ghost on the lattice $\rightarrow$ Landau-B-gauges, where B is related to the ghost dressing function at zero momentum.

IR behavior of the maximally Abelian gauge

Lattice calculations [e. g. Mendes et al., 0809.3741] and the refined GZ framework [Capri et al., PRD 77 (2008)] support a decoupling scenario (all propagators finite).

Is there a scaling solution even possible?

An investigation using functional methods is also desirable from other points of view:

- Calculations on lattice and in refined GZ framework done in $SU(2)$ $\rightarrow$ generalization to $SU(N)$?
- IR region easier accessible by continuum methods than by lattice calculations.

Can the methods used in the Landau gauge be applied straightforwardly?

Landau gauge and maximally Abelian gauge: Current Status

	Landau gauge	MAG
propagators	A, c	A, B, c
interactions	AAA, Acc; $AAAA$	ABB, Acc; $AABB$, $AAcc$, $BBcc$, $BBBB$, $cccc$
Gribov region	bounded in all directions	bounded in off-diagonal and unbounded in diagonal direction [Capri et al, PRD 79 (2009)]
decoupling sol.	lattice, refined Gribov-Zwanziger framework and functional equations	lattice [Mendes et al., arXiv:0809.3741], ref. Gribov-Zwanziger framework [Capri et al., PRD 77 (2008)]; $SU(2)$ only
scaling solution	funct. eqs., lattice in 2d [von Smekal, Alkofer, Hauck, PRL 79 (1997); Pawlowski et al., PRL 93 (2004); Maas, PRD 75 (2007)]	this talk ($SU(N)$) [M.Q.H., Schwenzer, Alkofer, arXiv:0904.1873]

The maximally Abelian gauge (MAG)

Dual superconductor picture of confinement (Mandelstam, 't Hooft)

Magnetic monopoles condense $\longrightarrow$ squeeze electric flux into flux tubes.

Ezawa, Iwazaki, PRD 25 (1981): Hypothesis of Abelian dominance (Abelian part should dominate the infrared part of the theory, since monopoles live in Abelian part of algebra.)

Suzuki et al., 0907.0583

- String tension is the same from non-Abelian, Abelian and monopole part, if no gauge is fixed.
- Colorelectric flux is squeezed into flux tubes.
- $\Rightarrow$ Confinement by monopoles.

String tensions of Abelian and monopole part agree more or less with the string tension of the non-Abelian part, if MAG is employed. $\Rightarrow$ MAG a "cheap" way to obtain monopoles?

Diagonal and off-diagonal fields

Gauge field components:

$$A_\mu = A^i_\mu T^i + B^a_\mu T^a, \quad i = 1, \ldots, N-1, \quad a = N, \ldots, N^2-1$$

Abelian subalgebra: $[T^i, T^j] = 0$, can be written as diagonal matrices
$\Rightarrow$ Abelian/diagonal fields $\boldsymbol{A}$, non-Abelian/off-diagonal fields $\boldsymbol{B}$.

E.g. $T^1 = \frac{1}{2}\lambda^3$, $T^2 = \frac{1}{2}\lambda^8$ for $SU(3)$.

$[\boldsymbol{T^r}, \boldsymbol{T^s}] = \boldsymbol{i\, f^{rst}\, T^t}$

$$f^{ijk} = 0, \quad f^{ija} = 0, \quad f^{iab} \neq 0$$

$$SU(2): \quad f^{abc} = 0, \qquad SU(N>2): \quad f^{abc} \neq 0$$

$SU(2)$: only 2 off-diagonal and 1 diagonal fields $\Rightarrow$ only 1 possible set of field combinations for three-point function
$SU(N>2)$: three off-diagonal fields can interact

Gauge fixing condition of the MAG

Stress role of diagonal fields $\Rightarrow$ minimize norm of off-diagonal field $\boldsymbol{B}$:

$$||B_U|| = \int dx\, B_U^a B_U^a \rightarrow \text{minimize wrt. gauge transformations } U$$

$D_\mu^{ab} \boldsymbol{B}_\mu^b = (\delta_{ab}\partial_\mu - g f^{abi} \boldsymbol{A}_\mu^i) \boldsymbol{B}_\mu^b = 0$ non-linear gauge fixing condition!

Remaining symmetry of diagonal part: $U(1)^{N-1}$
Fix gauge of diag. gluon field $\boldsymbol{A}$ by Landau gauge condition: $\partial_\mu \boldsymbol{A}_\mu = 0$.

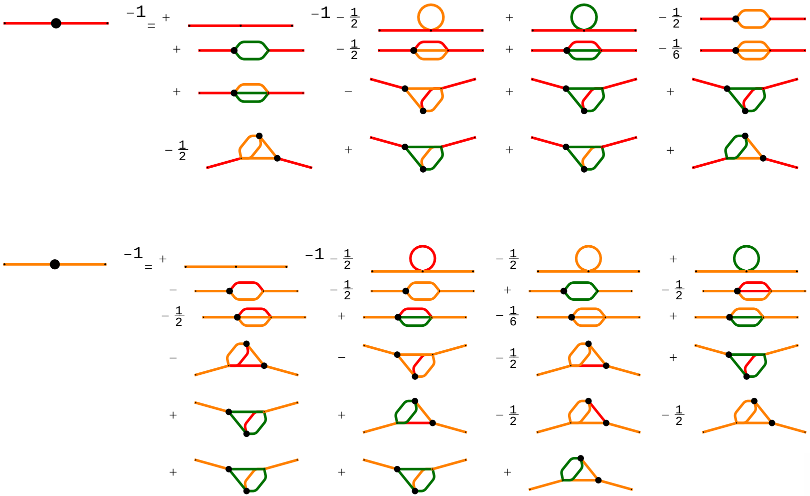
Peculiarities of the maximally Abelian gauge for $SU(2)$

- Yang-Mills vertices split: ***ABB***, ***AABB***, ***BBBB***.
- Non-linear gauge fixing condition (depends on ***A***) $\rightarrow$ ***Acc***, ***AAcc***, ***BBcc***.
- Renormalizability requires an additional quartic ghost interaction $\rightarrow$ ***cccc***.
- Ghosts also split into diagonal and off-diagonal parts, but diagonal ghosts decouple (diagonal ghost equation).
- Two gauge fixing parameters: $\alpha_A = 0$ (Landau gauge), α_B.

Note: For $SU(N>2)$ there are more interactions (due to f^{abc}): ***BBB***, ***Bcc***, ***ABBB***, ***ABcc*** $\rightarrow$ more DSEs with more terms.

This plethora of interactions makes the equations much more intricate than in Landau gauge. To consider all possible solutions an improved method is necessary.

DSEs of the MAG



Loop integrals for low external momenta

We want to know how a vertex function behaves, when the external momenta approach 0 simultaneously: $\Gamma(p_1, p_2, \ldots)$ for $p_i \to 0$

kinematic divergences:
$\Gamma(p_1, p_2, \ldots)$ for $p_1 \to 0$, $p_2, \ldots = const$ [Alkofer, M.Q.H., Schwenzer, EPJC 62 (2009)]

Generic propagator

$$L_{(\mu\nu)} \cdot \frac{D(p)}{p^2},$$

assume power law behavior at low p

$$D^{IR}(p) = A \cdot (p^2)^\delta$$

IR exponent

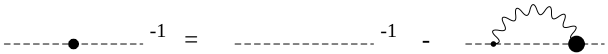
Integrals are dominated by $1/(p-q)^2 \to$ use IR expressions for all quantities.

Vertices also assume power law behavior [Alkofer, Fischer, Llanes-Estrada, PLB 611 (2005) (skeleton expansion)].

In scaling solutions the qualitative behavior of the whole tower of equations can be determined.

Power counting

- The ghost propagator DSE:

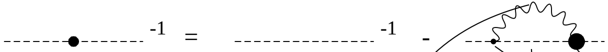


- Plug in power law ansätze for dressing functions in the IR (In Landau gauge the ghost-gluon vertex has an IR constant dressing.):

$$\left(\frac{B\cdot(p^2)^{\beta}}{p^2}\right)^{-1} \sim \int \frac{d^d q}{(2\pi)^d}\, P_{\mu\nu}\, \frac{A\cdot(q^2)^{\alpha}}{q^2}\, \frac{B\cdot((p-q)^2)^{\beta}}{(p-q)^2}\, (p-q)_{\mu} q_{\nu}$$

Power counting

- The ghost propagator DSE:



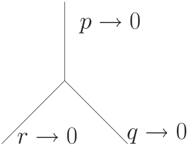
- Plug in power law ansätze for dressing functions in the IR (In Landau gauge the ghost-gluon vertex has an IR constant dressing.):

$$\left(\frac{B\cdot(p^2)^{\beta}}{p^2}\right)^{-1} \sim \int \frac{d^d q}{(2\pi)^d}\, P_{\mu\nu}\, \frac{A\cdot(q^2)^{\alpha}}{q^2}\, \frac{B\cdot((p-q)^2)^{\beta}}{(p-q)^2}\, (p-q)_{\mu} q_{\nu}$$

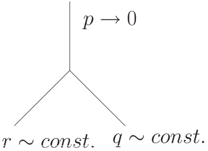
- Only one momentum scale
  $\longrightarrow$ simple power counting is possible $\longrightarrow$ scaling relation:

- $1-\beta = \frac{d}{2} + \alpha - 1 + \beta - 1 + \frac{1}{2} + \frac{1}{2} \Longrightarrow -2\beta = \alpha + \frac{d}{2} - 2$

More infrared exponents



Up to now it was assumed that all momenta of a Green function go to zero simultaneously $\to$ uniform IR exponents.



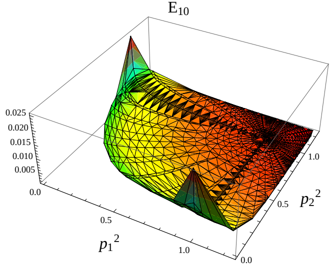
For a three-point function there is one additional possibility: Only one momentum goes to zero $\to$ kinematic IR exponents [Alkofer, M.Q.H., Schwenzer, 0801.2762].

There is a non-uniform dependence on the momenta [Alkofer, M.Q.H., Schwenzer, EPJC 62 (2009)]; ex. on next slide.

More infrared exponents: Examples

The first order contribution of the three-gluon vertex in the IR (ghost triangle) can be described by 10 dressing functions:

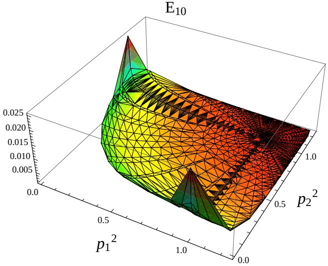
$$\Gamma^{\Delta}_{\mu\nu\rho}(p_1,p_2,p_3) = \sum_{i=1}^{10} E_i(p_1^2,p_2^2,p_3^2;\kappa;d)\tau^i_{\mu\nu\rho}(p_1,p_2,p_3).$$

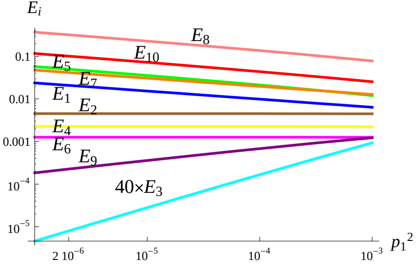


More infrared exponents: Examples

The first order contribution of the three-gluon vertex in the IR (ghost triangle) can be described by 10 dressing functions:

$$\Gamma^{\Delta}_{\mu\nu\rho}(p_1,p_2,p_3)=\sum_{i=1}^{10}E_i(p_1^2,p_2^2,p_3^2;\kappa;d)\tau^i_{\mu\nu\rho}(p_1,p_2,p_3).$$





These singularities only appear in the longitudinal parts [Alkofer, M.Q.H., Schwenzer, EPJC 62 (2009)] $\Rightarrow$ play no role in Landau gauge [Fischer, Maas, Pawlowski, 0810.1987; Fischer, Pawlowski, PRD 80 (2009)].

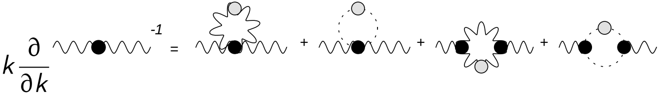
Functional renormalization group equations

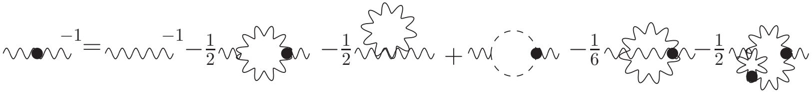
Functional equations similar to DSEs, but with decisive differences:

- only 1-loop diagrams
- ALL quantities dressed
- (appearance of regulator)

Renormalization group equations (RGEs) are "differential DSEs".

Compare RGE and DSE of gluon propagator:





Deriving a scaling relation

DSE-FRG consistency condition by Fischer & Pawlowski

[Fischer, Pawlowski, PRD 75 (2005)]

- Investigated system of DSEs/RGEs in Landau gauge.
- The ghost propagator DSE has only 1 loop diagram.
- The ghost propagator RGE has 4 loop diagrams.
- For consistent solutions of DSEs and RGEs you expect the same scaling of the propagators $\Rightarrow$ the DSE diagram has to match the counting of the RGE diagrams.
- Connection between DSEs and RGEs is the bare ghost-gluon vertex, which is not IR enhanced.

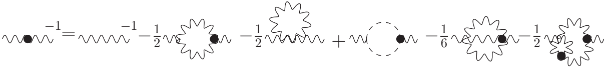
We proof here in general [M.Q.H., Schwenzer, Alkofer, 0904:1873]:

Scaling relations are intimately connected to the appearance of bare vertices in DSEs and not in RGEs. $\Rightarrow$ Possible scaling relations can be read off from the interactions.

System of inequalities

- For every diagram the IR can be written down.
- At least the IRE of one diagram must equal the IRE of the vertex function on the lhs.
- No diagram can be more IR divergent than the vertex function on the lhs $\rightarrow \delta_{lhs} \leq \delta_{rhs}$.
- Not knowing which diagram is leading on the rhs, we can write inequalities from all diagrams.

$$-\delta_{gl} = \min(\underbrace{0}_{\text{bare prop.}}, \underbrace{2\delta_{gh}+\delta_{gg}}_{\text{gl loop}}, \underbrace{\delta_{gl}}_{\text{tadpole}}, \underbrace{2\delta_{gl}+\delta_{3g}}_{\text{gh loop}}, \underbrace{3\delta_{gl}+\delta_{4g}}_{\text{sunset}}, \underbrace{4\delta_{gl}+2\delta_{3g}}_{\text{squint}})$$



Relevant inequalities

A closed form for all relevant inequalities can be derived from DSEs and RGEs.
2 types:

type		derived from	#
dressed vertices	$C_1 := \delta_{vertex} + \frac{1}{2} \sum_{\substack{\text{legs } j \text{ of} \\ \text{vertex}}} \delta_j \geq 0$	RGEs	infinite
prim. div. vertices	$\boldsymbol{C_2 := \frac{1}{2} \sum_{\substack{\text{legs } j \text{ of} \\ \text{prim. div.} \\ \text{vertex}}} \delta_j \geq 0}$	DSEs/RGEs	finite

Some inqualities are contained within others.
E. g. in MAG: $\delta_B \geq 0$ and $\delta_c \geq 0$ render $\delta_B + \delta_c \geq 0$ useless.

NB: These inequalities explicitly show that the skeleton expansion used in previous studies is a consistent expansion. However, the skeleton expansion is now obsolete.

Infrared exponent for an arbitrary diagram

Having so many diagrams, isn't there a shorter way than writing all expressions down explicitly?

Arbitrary Diagram v

Numbers of vertices and propagators related $\Rightarrow$ possible to get a formula for the IR exponent by pure combinatorics in terms of:

- propagator IR exponents δ_{ϕ_i}
- number of vertices
- number of external legs m^{ϕ_i}

$$\delta_v = \boxed{-\frac{1}{2}\sum_i m^i \delta_i} + \\ + \sum_i (\text{\# of dressed vertices})_i C_1^i + \sum_i (\text{\# of bare vertices})_i C_2^i$$

Infrared exponent for an arbitrary diagram

Having so many diagrams, isn't there a shorter way than writing all expressions down explicitly?

Arbitrary Diagram v

Numbers of vertices and propagators related $\Rightarrow$ possible to get a formula for the IR exponent by pure combinatorics in terms of:

- propagator IR exponents δ_{ϕ_i}
- number of vertices
- number of external legs m^{ϕ_i}

lower bound on IRE

$$\delta_v = \boxed{-\frac{1}{2}\sum_i m^i \delta_i} + \\ + \sum_i (\#\text{ of dressed vertices})_i C_1^i + \sum_i (\#\text{ of bare vertices})_i C_2^i$$

Only depends on the external legs $\rightarrow$ equal for all diagrams in a DSE/RGE [M.Q.H., Schwenzer, Alkofer, arXiv:0904.1873].

[Similar formula with slightly different arguments: Fischer, Pawlowski, arXiv:0903.2193]

Scaling relations

General analysis of propagator DSEs

At least one inequality from a prim. divergent vertex has to be saturated, i. e. $\boxed{C_2^i = 0 \text{ for at least one } i}$.

Necessary condition for a scaling solution.

Related to bare vertices in DSEs: Fischer-Pawlowski consistency condition DSEs $\leftrightarrow$ RGEs [Fischer, Pawlowski, PRD 75 (2007)].

$\Rightarrow$ One primitively divergent vertex is not IR enhanced.

This does not necessarily mean that it is bare:

- Dependence on momentum configuration.
- Consider different dressing functions: Vanishing or constant.

The non-enhanced vertex is also called the leading vertex, because it determines the leading diagram in a DSE.

The non-enhancement of at least one primitively divergent vertex is now established for all scaling type solutions. [M.Q.H., Schwenzer, Alkofer, arXiv:0804.1873]

How to obtain a scaling relation: Landau gauge

1. Look at all inequalities for primitively divergent vertices, i. e. at C_2^i.
2. Try all possibilities of $C_2^i = 0$.
3. Choose the non-trivial solutions.

How to obtain a scaling relation: Landau gauge

1. Look at all inequalities for primitively divergent vertices, i. e. at C_2^i.
2. Try all possibilities of $C_2^i = 0$.
3. Choose the non-trivial solutions.

Application to Landau gauge:

1. $\delta_{gl} \geq 0$, $\delta_{gl} + 2\delta_{gh} \geq 0$
2. a $\delta_{gl} = 0$
   b $\delta_{gl} + 2\delta_{gh} = 0$
3. a $\cancel{\delta_{gl} = \delta_{gh} = 0}$
   b $\delta_{gl} + 2\delta_{gh} = 0$

Scaling relation of the Landau gauge: $\boxed{\frac{1}{2}\delta_{gl} = -\delta_{gh} = \kappa_{LG}}$

From scaling relation to vertices

How to get the IRE of an arbitrary vertex?

1. Start with an appropriate propagator DSE.
2. Add successively the leading vertex until you get the desired vertex.

A general formula for m gluon and $2n$ ghost legs in d dimensions can be determined [Alkofer, Fischer, Llanes-Estrada, PLB 611 (2005); M.Q.H., Alkofer, Fischer, Schwenzer, PLB 659 (2008)]:

$$\delta_{m,2n} = (n-m)\,\kappa_{LG} + (1-n)\left(\frac{d}{2}-2\right)$$

How to obtain a scaling relation: MAG

Many interactions ⇒ many inqualities, but some of them are contained within others ⇒ reduces number of possibilities.

1. Look at all inequalities for primitively divergent vertices, i. e. at C_2^i.
2. Try all possibilities of $C_2^i = 0$.
3. Choose the non-trivial solutions.

How to obtain a scaling relation: MAG

Many interactions ⇒ many inqualities, but some of them are contained within others ⇒ reduces number of possibilities.

1. Look at all inequalities for primitively divergent vertices, i. e. at C_2^i.
2. Try all possibilities of $C_2^i = 0$.
3. Choose the non-trivial solutions.

Application to the MAG:

1. $\delta_B \geq 0,\ \delta_c \geq 0,\ \delta_A + \delta_B \geq 0,\ \delta_A + \delta_c \geq 0$
2. a $\delta_B = 0$
   b $\delta_c = 0$
   c $\delta_A + \delta_B = 0$
   d $\delta_A + \delta_c = 0$
3. a $\cancel{\delta_A = \delta_B = \delta_c = 0}$
   b $\cancel{\delta_A = \delta_B = \delta_c = 0}$
   c $\delta_A + \delta_B = 0$
   d $\delta_A + \delta_c = 0$

Scaling relation of the MAG: $\boxed{\delta_B = \delta_c = -\delta_A = \kappa_{MAG}}$

The MAG in $SU(3)$

In general $SU(N)$ there are more interactions than included above.
$\rightarrow$ Different solution for "physical system", i. e. $SU(3)$?

4 additional vertices: ***BBB***, ***Bcc***, ***ABBB***, ***ABcc***
Constraints:

$$\frac{3}{2}\delta_B \geq 0, \qquad \frac{1}{2}\delta_B + \delta_c \geq 0,$$

$$\frac{1}{2}\delta_A + \frac{3}{2}\delta_B \geq 0, \qquad \frac{1}{2}\delta_A + \frac{1}{2}\delta_B + \delta_c \geq 0$$

Solution for $SU(N > 2)$ = solution for $SU(2)$

- Constraints already contained in "old" system $\rightarrow$ nothing new, solution still valid.
- No new solutions possible $\rightarrow$ unique solution.

IR Scaling solutions for other gauges

The analysis can be used also for other gauges. Beware: This corresponds to a naive application!

Linear covariant gauges	Ghost-antighost symmetric gauges
scaling solution only, if the longitudinal part of the gluon propagator gets dressed, but gauge fixing condition $\Rightarrow$ longitudinal part bare	quartic ghost interaction $\rightarrow \delta_{gh} \geq 0$ $\rightarrow$ with non-negative IREs only the trivial solution can be realized

This is valid for all possible dressings and agrees with the results from [Alkofer, Fischer, Reinhardt, v. Smekal, PRD 68 (2003)], where only certain dressings were considered.

$\Longrightarrow$

- Either the existence of a scaling solution is something special (?) or
- a more refined analysis is needed in these cases.

IR propagators of the MAG

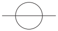
$$-\delta_A = \delta_B = \delta_c =: \kappa \geq 0$$

The scaling solution for the MAG differs in several qualitative and technical aspects from the Landau gauge solution:

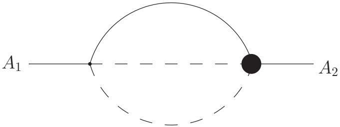
- Diagonal gluon propagator is IR enhanced ($\delta_A \leq 0$). $\Rightarrow$ Supports hypothesis of Abelian dominance.
- Off-diagonal propagators are IR suppressed.
- Two-loop terms are leading.
- Different structure of IR leading terms $\rightarrow$ new method for numerical solutions required.
- Different DSEs for $SU(2)$ and $SU(3)$ $\rightarrow$ different solutions? $\rightarrow$ No.
- Maas, 0807.5185: Parameter equivalent to ghost propagator in Landau gauge is here the diagonal gluon propagator.
- Abelian configurations are on the gauge orbit of the configurations at the Gribov horizon in Landau gauge [Greensite, Olejník, Zwanziger, PRD 69 (2004)].

Higher vertex functions in the MAG

Leading diagrams are determined by bare ***AABB*** or ***AAcc*** vertices:

sunset 	squint 
leading	possibly leading

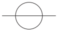
n-point functions (n even): Successively add pairs of fields:



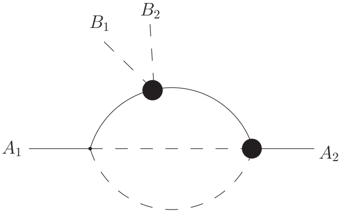
n odd: At least one vertex with an odd number of legs, cannot be determined uniquely (leading vertex is even; how to construct an odd vertex?)

Higher vertex functions in the MAG

Leading diagrams are determined by bare ***AABB*** or ***AAcc*** vertices:

sunset 	squint 
leading	possibly leading

n-point functions (n even): Successively add pairs of fields:



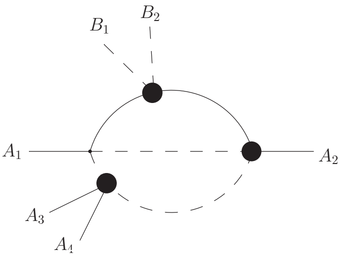
n odd: At least one vertex with an odd number of legs, cannot be determined uniquely (leading vertex is even; how to construct an odd vertex?)

Higher vertex functions in the MAG

Leading diagrams are determined by bare ***AABB*** or ***AAcc*** vertices:

sunset	squint
leading	possibly leading

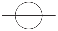
n-point functions (n even): Successively add pairs of fields:



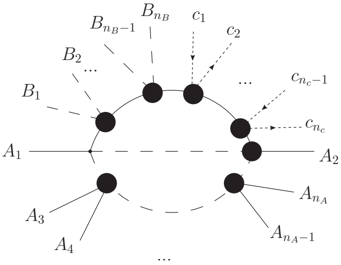
n odd: At least one vertex with an odd number of legs, cannot be determined uniquely (leading vertex is even; how to construct an odd vertex?)

Higher vertex functions in the MAG

Leading diagrams are determined by bare ***AABB*** or ***AAcc*** vertices:

sunset 	squint 
leading	possibly leading

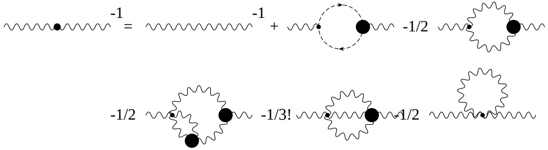
n-point functions (n even): Successively add pairs of fields:



n odd: At least one vertex with an odd number of legs, cannot be determined uniquely (leading vertex is even; how to construct an odd vertex?)

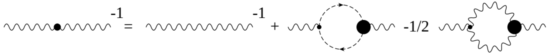
Numerical solution

In Landau gauge truncation "straightforward": keep one-loop terms (consistent UV behavior, contain IR leading term)



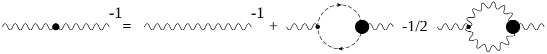
Numerical solution

In Landau gauge truncation "straightforward": keep one-loop terms (consistent UV behavior, contain IR leading term)

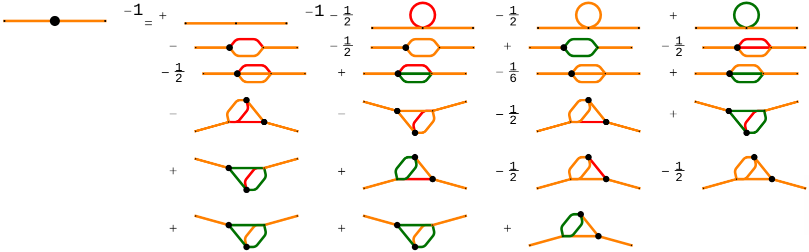


Numerical solution

In Landau gauge truncation "straightforward": keep one-loop terms (consistent UV behavior, contain IR leading term)



In MAG: two-loop terms leading $\longrightarrow$ for consistent UV behavior keep ALL two-loop terms = no truncation



Complete solution for the MAG

- Truncation?
- Odd vertices?
- The IR part has to connect to the mid-momentum and UV-part found by a numerical calculation.
- Due to the involved structure of the terms there is ample space for delicate cancellations (cf. propagator DSEs in Landau gauge: 1 diagram IR leading).
- Considerable more construction work for the tensors of the leading vertices (four-point functions: color $\times$ Lorentz $= 3 \times 10$ and 3×138) is necessary than in Landau gauge (ghost-gluon vertex: 2).

Summary

- It was shown for general systems of functional equations how the Fischer-Pawlowski consistency condition can lead to a scaling solution.
- As expected (at least) one vertex does not get IR enhanced.
- Qualitative solution for whole tower of functional equations.
- Skeleton expansion, as used earlier, obsolete.
- High number of interactions can be handled, because it is not necessary to write down all equations explicitly.
- Derivation of method technical, but it allows a straightforward application based only on the type of interactions in the Lagrangian.
- Method allows a first assessment what a scaling solution might look like. $\longrightarrow$ Input for a complete numeric calculation.

Conclusions on MAG

- The MAG may possess an IR scaling solution.
- This solution is in support of the hypothesis of Abelian dominance, because the diagonal gluon propagator is IR enhanced and thereby the dynamics in the IR are dominated by the diagonal gluon.
- Relation to monopole condensation has to be clarified.
- Although the DSEs are more complicated for general $SU(N>2)$, the qualitative behavior is the same as in $SU(2)$.

The existence of the IR scaling solution in the MAG has to be verified by a numerical solution of the DSEs, which is more involved than in Landau gauge. $\rightarrow$ Task for the future.