Quantum chromodynamics from the functional point of view

Markus Q. Huber

Institute of Physics, University of Graz

Seminar at APC, Université Paris Diderot

FWF Der Wissenschaftsfonds.

INSTITUT FÜR PHYSIK

Outline

The world in terms of particles

The standard model:

Quarks	d	s	b	g
	u	c	t	γ
Leptons	e	μ	τ	Z
	ν_e	ν_μ	ν_τ	$W^\pm$

Bosons

Higgs?

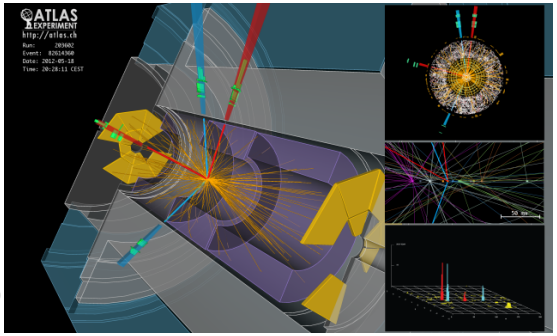
The world in terms of particles

The standard model:

Quarks	d	s	b	g
	u	c	t	γ
Leptons	e	μ	τ	Z
	ν_e	ν_μ	ν_τ	$W^\pm$

Bosons

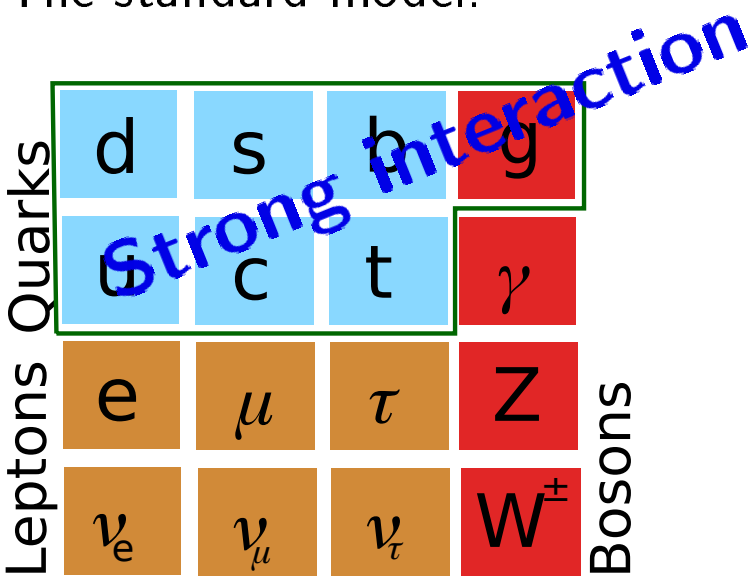
Higgs? 2012 Which Higgs?



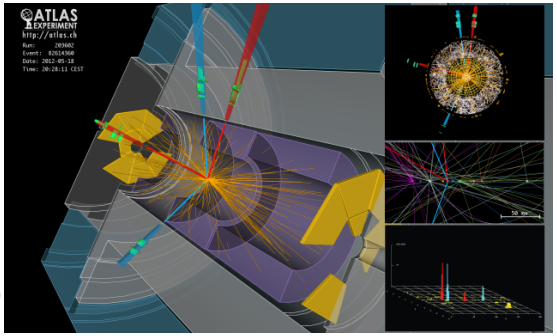
©CERN

The world in terms of particles

The standard model:



Higgs? 2012 Which Higgs?

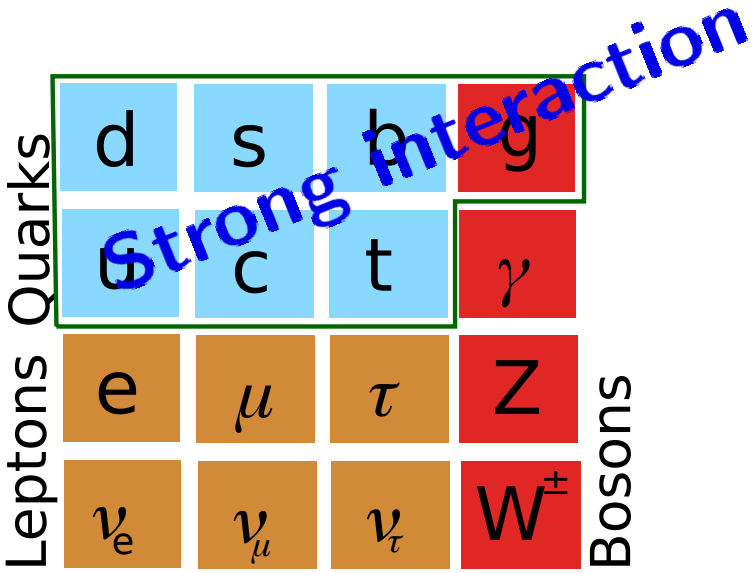


©CERN

Quantum chromodynamics:
self-consistent by itself, could be even fundamental

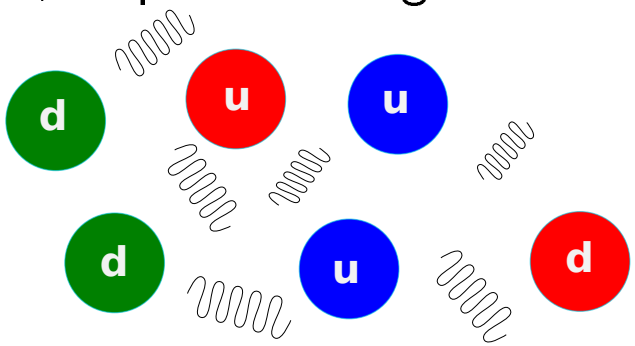
The world in terms of particles

The standard model:

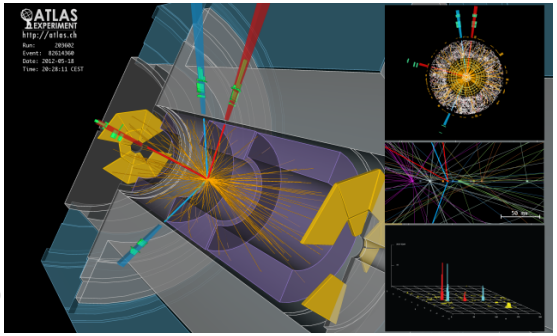


fundamental fields: quarks and gluons

u, d quarks are light $\sim$ MeV

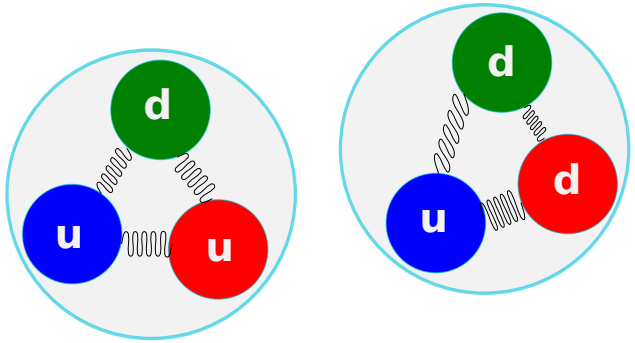


Higgs? 2012 Which Higgs?



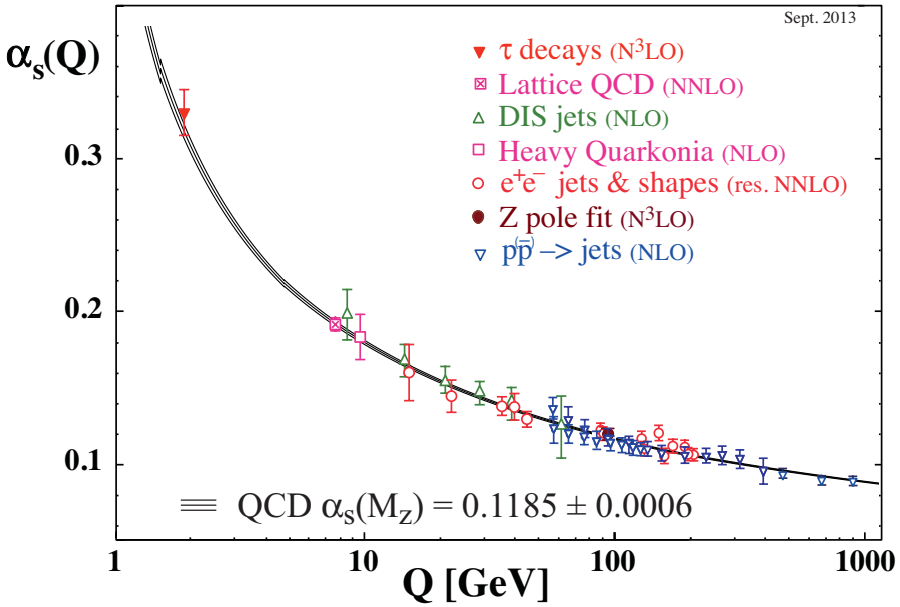
©CERN

physical degrees of freedom: hadrons $\sim$ GeV



How to investigate QCD

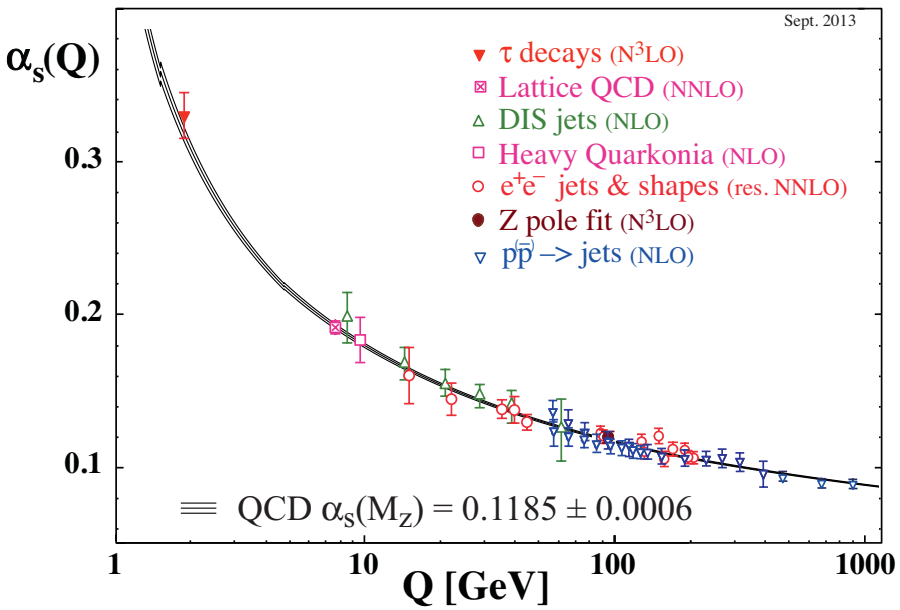
Asymptotic freedom (Nobel prize 2004):



Perturbative description at high energies.
Plenty of applications.

How to investigate QCD

Asymptotic freedom (Nobel prize 2004):



Perturbative description at high energies.
Plenty of applications.

- Perturbative series is not convergent.
- Non-perturbative phenomena?
  - E.g., no mass creation to every order in perturbation theory.

$\Rightarrow$ Non-perturbative methods required.

Perturbation theory based on non-perturbative 'models', e.g.,

- (Refined) Gribov-Zwanziger model
- Massive extension [Peláez, Reinosa, Serreau, Tissier, Tresmontant, Wschebor, '10-'14]

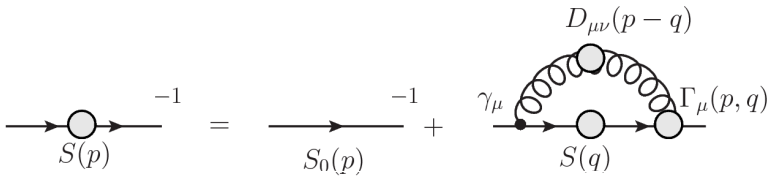
The family of functional equations

Coupled integro-differential/integral equations.

The family of functional equations

Coupled integro-differential/integral equations.

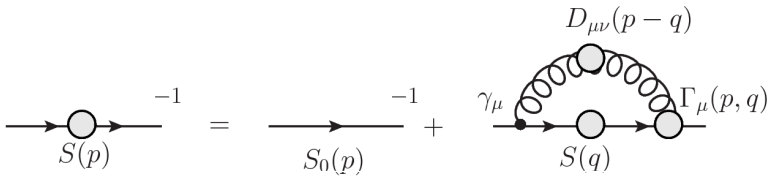
- Dyson-Schwinger equations: eqs. of motion for correlation functions



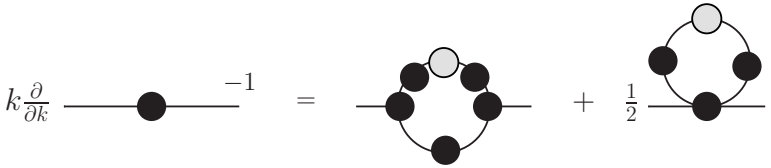
The family of functional equations

Coupled integro-differential/integral equations.

- Dyson-Schwinger equations: eqs. of motion for correlation functions



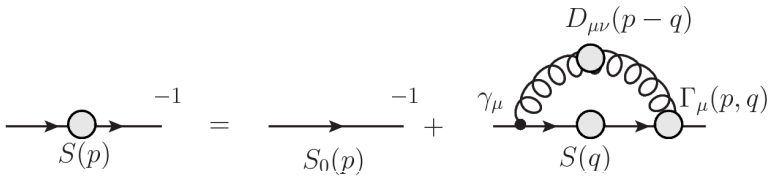
- Functional renormalization group: flow equations, RG scale k, regulator



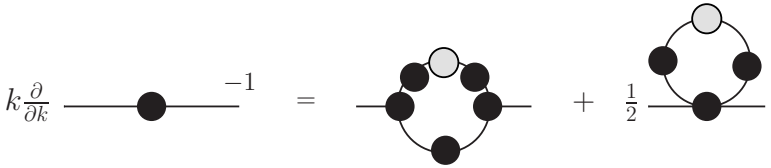
The family of functional equations

Coupled integro-differential/integral equations.

- Dyson-Schwinger equations: eqs. of motion for correlation functions



- Functional renormalization group: flow equations, RG scale k, regulator

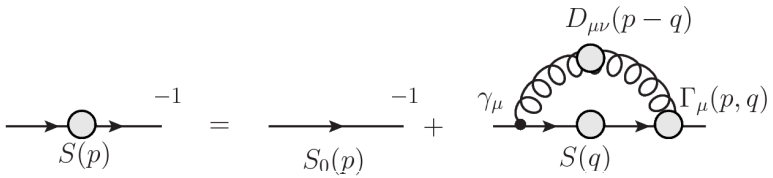


- N-PI effective action

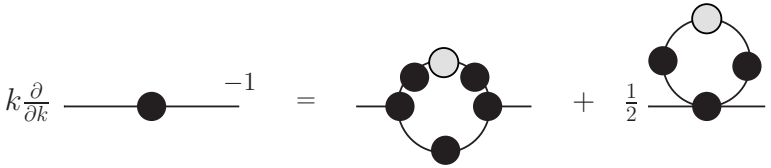
The family of functional equations

Coupled integro-differential/integral equations.

- Dyson-Schwinger equations: eqs. of motion for correlation functions



- Functional renormalization group: flow equations, RG scale k, regulator



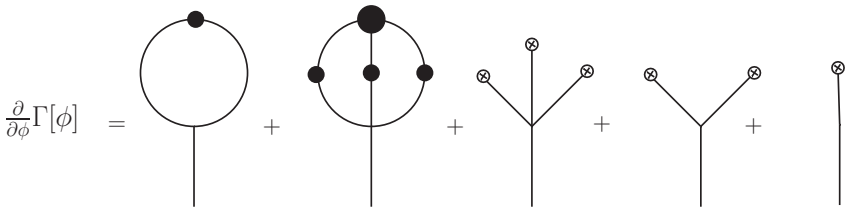
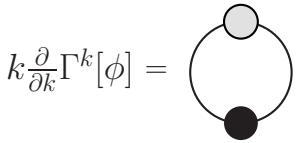
- N-PI effective action

Non-perturbative in the sense:

- Exact equations.
- No small coupling required.

In reality they cannot be solved exactly (with a few exceptions).
Self-consistence!

Comparison: DSEs and flow equations

Dyson-Schwinger equations (DSEs)	Functional RG equations (FRGEs)
'integrated flow equations'	'differential DSEs'
effective action $\Gamma[\phi]$	effective average action $\Gamma^k[\phi]$
–	regulator
n-loop structure (n *const.*)	1-loop structure
exactly only bare vertex per diagram	no bare vertices
	

- Both systems of equations are exact.
- Both contain infinitely many equations.

From Green functions to 'observables'

Functional equations are expressed in terms of
Green functions/correlation functions/n-point functions.

The effective action is the generating functional of 1PI Green functions. $\longleftrightarrow$ The set of **all** Green functions describes the theory completely.

Green functions $\rightarrow$ 'observables'?

Examples:

- Bound state equations $\rightarrow$ masses and properties of hadrons
- Analytic properties of Green functions $\rightarrow$ confinement
- (Pseudo-)Order parameters

Functional equations and lattice methods

	functional equations	lattice
source of error	truncation	finite volume, finite lattice spacing, statistics
temperature	✓	✓
chemical potential	✓	sign problem
analytic structure	✓	no

Functional equations and lattice methods

	functional equations	lattice
source of error	truncation	finite volume, finite lattice spacing, statistics
temperature	✓	✓
chemical potential	✓	sign problem
analytic structure	✓	no

Green functions	functional equations	lattice
propagators	✓	✓
three-point functions	ghost-gluon vertex: ✓ 3-gluon vertex: ✓ quark-gluon vertex: (✓)	limited mom. dependence
four-point functions	✓	not soon

Dyson-Schwinger/Schwinger-Dyson-equations

1. Start from path integral: Integral of derivative vanishes.

$$0 = \int D[\phi] \frac{\delta}{\delta \phi} e^{-S + \int dy \phi(y) J(y)}$$

Details and example of scalar theory:
`http://tinyurl.com/dsenotes`

Dyson-Schwinger/Schwinger-Dyson-equations

1. Start from path integral: Integral of derivative vanishes.

$$0 = \int D[\phi] \frac{\delta}{\delta\phi} e^{-S + \int dy \phi(y) J(y)}$$

2. Go to effective action $\Gamma[\phi_{\text{cl}}]$ (Legendre transform of $W[J] = \ln Z[J]$).

Details and example of scalar theory:
`http://tinyurl.com/dsenotes`

Dyson-Schwinger/Schwinger-Dyson-equations

1. Start from path integral: Integral of derivative vanishes.

$$0 = \int D[\phi] \frac{\delta}{\delta\phi} e^{-S + \int dy \phi(y) J(y)}$$

2. Go to effective action $\Gamma[\phi_{\text{cl}}]$ (Legendre transform of $W[J] = \ln Z[J]$).
3. Master equation:

$$\left.\frac{\delta S}{\delta\phi(x)}\right|_{\phi(x') = \phi_{\text{cl}}(x') + \int dz\, D(x',z)^J\, \delta/\delta\phi_{\text{cl}}(z)} = \frac{\delta\Gamma[\phi_{\text{cl}}]}{\delta\phi_{\text{cl}}(x)}$$

Details and example of scalar theory:
`http://tinyurl.com/dsenotes`

Dyson-Schwinger/Schwinger-Dyson-equations

1. Start from path integral: Integral of derivative vanishes.

$$0 = \int D[\phi] \frac{\delta}{\delta\phi} e^{-S + \int dy \phi(y) J(y)}$$

2. Go to effective action $\Gamma[\phi_{\text{cl}}]$ (Legendre transform of $W[J] = \ln Z[J]$).
3. Master equation:

$$\left.\frac{\delta S}{\delta\phi(x)}\right|_{\phi(x') = \phi_{\text{cl}}(x') + \int dz\, D(x',z)^J\, \delta/\delta\phi_{\text{cl}}(z)} = \frac{\delta\Gamma[\phi_{\text{cl}}]}{\delta\phi_{\text{cl}}(x)}$$

4. DSEs for Green functions by differentiating wrt fields.

Details and example of scalar theory:
`http://tinyurl.com/dsenotes`

Automated derivation

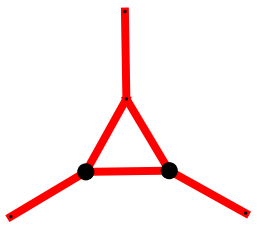
Derivation by hand becomes tedious:

- Large Lagrangians.
- Higher Green functions.
- Larger truncations.
- Error-prone.

Automated derivation

Derivation by hand becomes tedious:

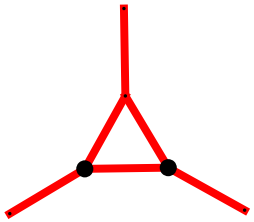
- Large Lagrangians.
- Higher Green functions.
- Larger truncations.
- Error-prone.



Automated derivation

Derivation by hand becomes tedious:

- Large Lagrangians.
- Higher Green functions.
- Larger truncations.
- Error-prone.

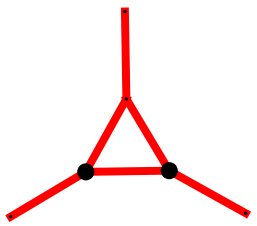


```
(2 g^2 Nc Z1 DAAA[y, qs + y + 2 sp[q, q1], (-y - sp[q, q1])/Sqrt[y (qs + y + 2 sp[q, q1])]]
  DAAA[x2 + y + 2 sp[p, q], qs + x2 - 2 sp[p, q1], (-x2 - sp[p, q] + sp[p, q1] + sp[q, q1])/Sqrt[(x2 + y + 2 sp[p, q]) (qs + x2 - 2 sp[p, q1])]] Dgl[qs] Dgl[qs + x2 - 2 sp[p, q1]] Dgl[
    sp[p, q]^4 (sp[p, q1]^2 sp[q, q1] (y + sp[q, q1]) + qs x2 (y (9 qs + 6 (x2 + y)) + (5 qs + 6 x2 + 10 y) sp[q, q1]) - sp[p, q1] (qs y (5 qs
    sp[p, q]^3 (2 sp[p, q1]^3 (qs y - sp[q, q1]^2) + sp[p, q1] (qs y (10 qs^2 + (-5 x2 - 3 y) y + qs (19 x2 + 3 y)) + (3 qs^3 + 8 qs x2 y + 21 qs^2 (
      qs x2 (y (-9 qs^2 + 3 x2^2 + 7 x2 y + 3 y^2 + 2 qs (x2 + y)) + (-10 qs^2 + qs (-3 x2 - 19 y) + x2 (3 x2 + 5 y)) sp[q, q1] + (-16 qs - 7 x2 - 11 y
      sp[p, q1]^2 (qs (-16 qs - 11 x2 - 7 y) y + (-5 qs^2 + qs (-9 x2 - 19 y) + 2 y (5 x2 + 3 y)) sp[q, q1] + (-5 qs + 12 (x2 + y)) sp[q, q1]^2 +
    sp[p, q]^2 (sp[p, q1]^4 sp[q, q1] (qs + sp[q, q1]) + sp[p, q1]^3 (qs y (7 qs + 11 x2 + 16 y) + (-6 qs^2 + y (9 x2 + 5 y) + qs (-10 x2 + 19 y)
      qs x2 (y (-3 qs^3 - 10 qs^2 (x2 + y) - 6 x2 y (x2 + y) + qs (-3 x2^2 - 19 x2 y - 3 y^2)) + (-6 qs^3 + qs^2 (-21 x2 - 32 y) + qs (-9 x2^2 - 60 x2 y
        (-15 qs^2 - 15 x2^2 + qs (-46 x2 - 41 y) - 41 x2 y - 12 y^2) sp[q, q1]^2 + (-7 qs - 16 x2 - 11 y) sp[q, q1]^3) + sp[p, q1]^2 (qs y (-15 q
        (3 qs^3 + qs^2 (5 x2 - 39 y) + qs (-81 x2 - 39 y) y + y^2 (5 x2 + 3 y)) sp[q, q1] + (12 qs^2 + 12 x2^2 + 3 x2 y + 12 y^2 + 3 qs (x2 + y)) sp[q
      sp[p, q1] (qs y (6 qs^3 + qs^2 (32 x2 + 21 y) + qs (25 x2^2 + 60 x2 y + 9 y^2) + x2 (3 x2^2 + 25 x2 y + 15 y^2)) + (15 qs^3 (x2 + y) + x2 y (-3 x2
        (-3 qs^3 + x2^2 (-3 x2 - 5 y) + qs^2 (39 x2 - 5 y) + qs x2 (39 x2 + 81 y)) sp[q, q1]^2 + (-6 qs^2 + qs (19 x2 - 10 y) + x2 (5 x2 + 9 y)) sp[
    x2 y (-sp[p, q1]^5 (qs + sp[q, q1]) + sp[p, q1]^4 (qs (6 qs + 6 x2 + 9 y) + (10 qs + 6 x2 + 5 y) sp[q, q1]) - qs (qs y - sp[q, q1]^2) (x2 (-
        (6 qs + 9 x2 + 6 y) sp[q, q1]^2 + sp[q, q1]^3) + sp[p, q1]^3 (qs (-3 qs^2 - 3 x2^2 + qs (-7 x2 - 2 y) - 2 x2 y + 9 y^2) + (-3 x2^2 + 3 x2 y +
      sp[p, q1]^2 (qs (-3 qs^2 (2 x2 + y) + qs (-6 x2^2 - 19 x2 y - 10 y^2) + y (-3 x2^2 - 10 x2 y - 3 y^2)) + (-3 qs^3 - 25 qs^2 (x2 + y) + qs (-15 x2
        (-12 qs^2 - 15 x2^2 - 46 x2 y - 15 y^2 - 41 qs (x2 + y)) sp[q, q1]^2 + (-11 qs - 16 x2 - 7 y) sp[q, q1]^3) +
```

Automated derivation

Derivation by hand becomes tedious:

- Large Lagrangians.
- Higher Green functions.
- Larger truncations.
- Error-prone.



- Automated derivation:
  Mathematica package *DoFun* [Alkofer, Huber, Schwenzer '08; Huber, Braun '11]
  `http://tinyurl.com/dofun2`
- Framework for numeric handling:
  C++ program *CrasyDSE* [Huber, Mitter '11]
  `http://tinyurl.com/crasydse`

Is it convergent?

Often people tend to think 'perturbatively':
No small parameter. $\rightarrow$ What means to control the calculation?

Is it convergent?

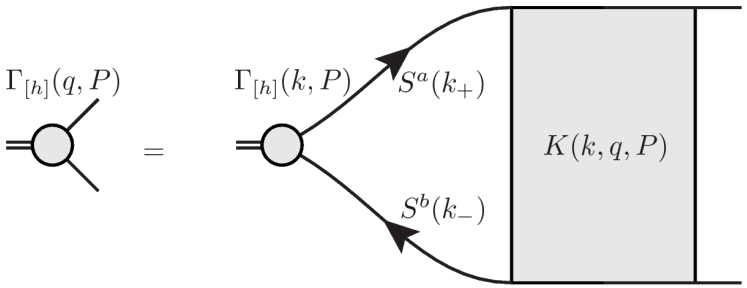
Often people tend to think 'perturbatively':
No small parameter. $\rightarrow$ What means to control the calculation?

- Comparisons with
  - perturbation theory
  - lattice calculation
- Sometimes analytic results possible.
- Deform truncation.

Will come back to this for Yang-Mills theory.

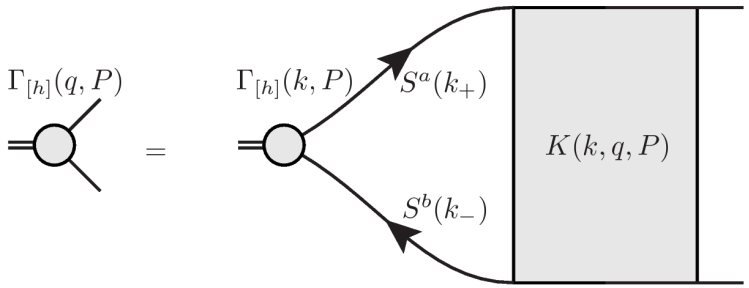
Bound state equations (BSE)

Example: Mesons from Bethe-Salpeter equation (BSE) with rainbow-ladder approximation

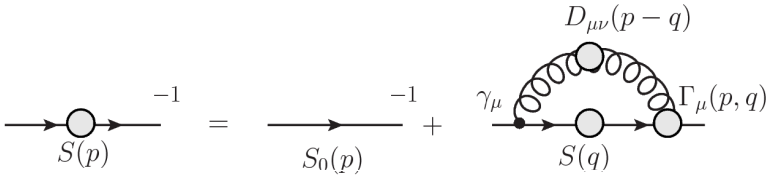


Bound state equations (BSE)

Example: Mesons from Bethe-Salpeter equation (BSE) with rainbow-ladder approximation

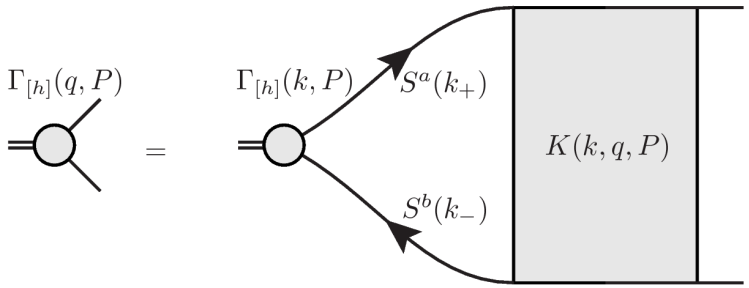


Contains quark propagator S and kernel K.

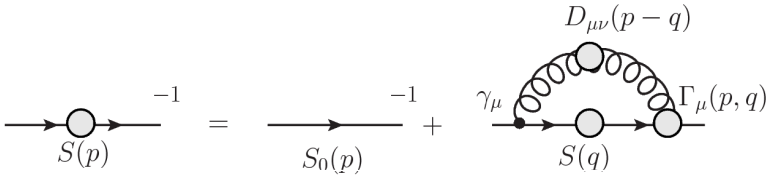


Bound state equations (BSE)

Example: Mesons from Bethe-Salpeter equation (BSE) with rainbow-ladder approximation



Contains quark propagator S and kernel K.



Rainbow-ladder

- $K \longrightarrow$ dressed one gluon exchange
- effective gluon propagator
- bare quark-gluon vertex

Rainbow-ladder approximation

What is special about rainbow-ladder?

Respects axial-vector Ward-Takahashi identity.
→ Consistent with chiral symmetry.
→ Pion is the massless Goldstone boson in the chiral limit.

Rainbow-ladder approximation

What is special about rainbow-ladder?

Respects axial-vector Ward-Takahashi identity.
→ Consistent with chiral symmetry.
→ Pion is the massless Goldstone boson in the chiral limit.

Dressed one-gluon exchange parametrized by model for the coupling:
Maris-Tandy '99, one effective parameter

$$\frac{G(p^2)}{p^2} = \frac{4\pi^2 D}{\omega^6} p^2 \exp^{-p^2/\omega^2} + 4\pi \frac{\gamma_m \pi F(p^2)}{1/2 \ln(\tau + (1 + p^2/\Lambda_{\mathrm{QCD}})^2)}$$

- Good description of many mesons with light quarks.

Rainbow-ladder approximation

What is special about rainbow-ladder?

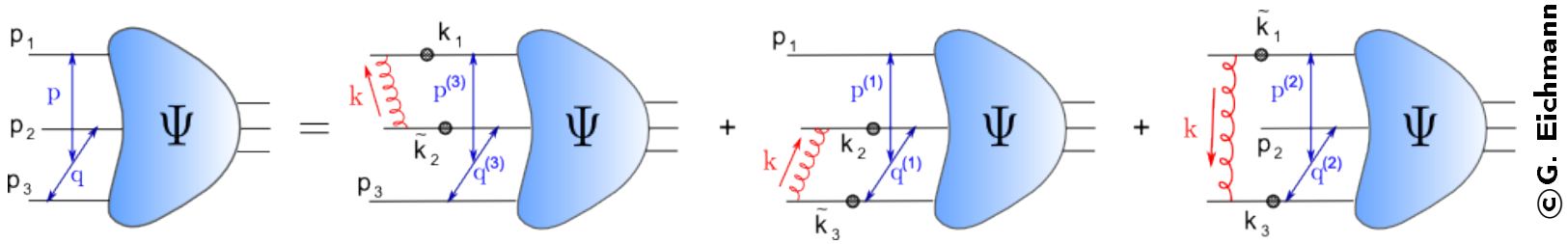
Respects axial-vector Ward-Takahashi identity.
→ Consistent with chiral symmetry.
→ Pion is the massless Goldstone boson in the chiral limit.

Dressed one-gluon exchange parametrized by model for the coupling:
Maris-Tandy '99, one effective parameter

$$\frac{G(p^2)}{p^2} = \frac{4\pi^2 D}{\omega^6} p^2 \exp^{-p^2/\omega^2} + 4\pi \frac{\gamma_m \pi F(p^2)}{1/2 \ln(\tau + (1 + p^2/\Lambda_{\mathrm{QCD}})^2)}$$

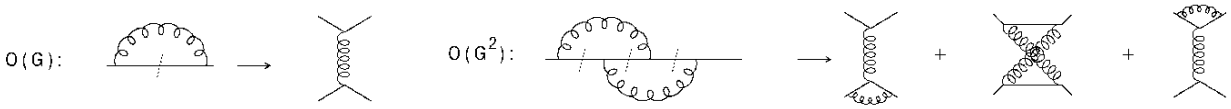
- Good description of many mesons with light quarks.

Baryons: Faddeev equation

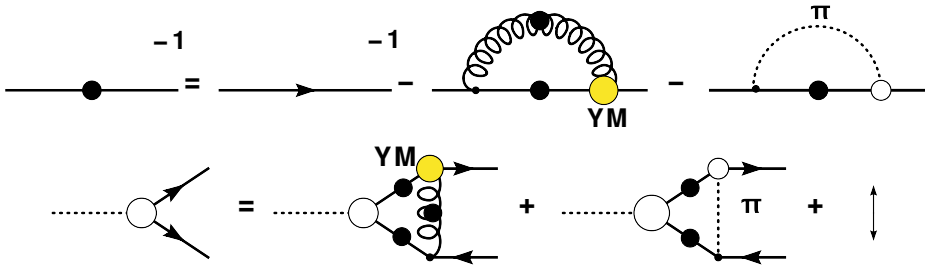


Beyond rainbow-ladder: Examples

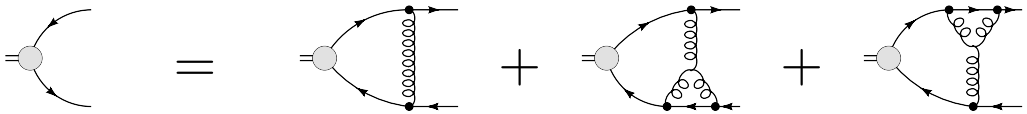
- Vertex dressing [Bender, Roberts, von Smekal '96]:



- Include pion back coupling effects [e.g., Fischer, Nickel, Wambach '07; Fischer, Nickel, Williams '08; Fischer, Williams '08]:



- Include gluon self-interaction [e.g., Maris, Tandy '06; Fischer, Williams '09]:

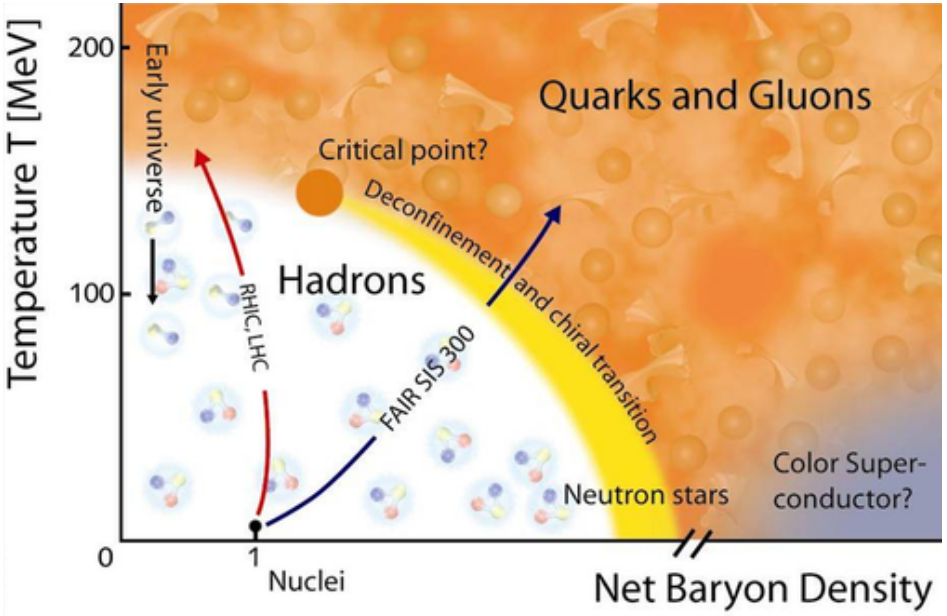


→ Three-gluon vertex required!

QCD phase diagram

Examples for accessible quantities:

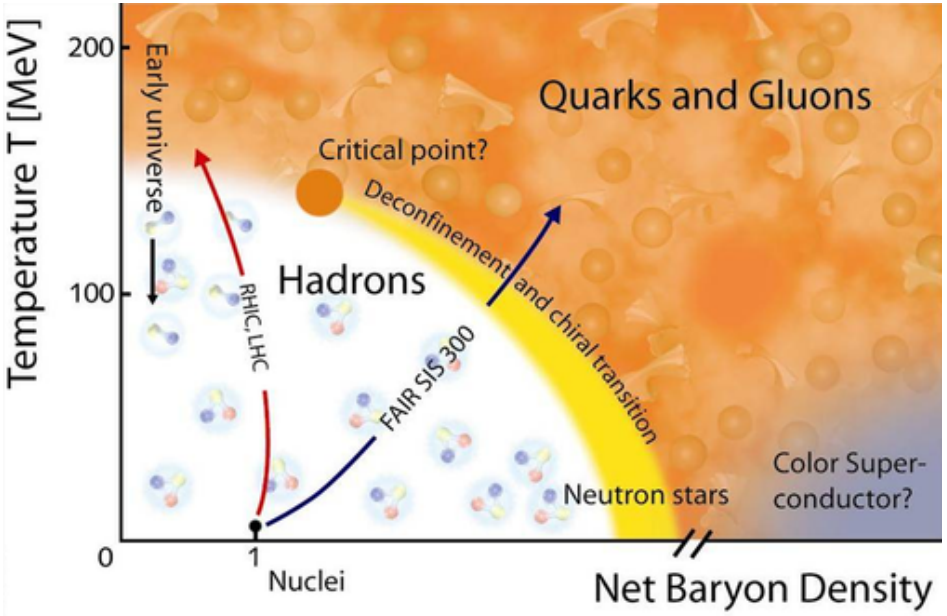
- quark condensate (chiral symm.)
- dual quark condensate/dressed Polyakov loop (conf.)
- dual density (conf.)
- (dual) quark dressing function (chiral symm./conf.)
- Polyakov loop potential (conf.)



QCD phase diagram

Examples for accessible quantities:

- quark condensate (chiral symm.)
- dual quark condensate/dressed Polyakov loop (conf.)
- dual density (conf.)
- (dual) quark dressing function (chiral symm./conf.)
- Polyakov loop potential (conf.)



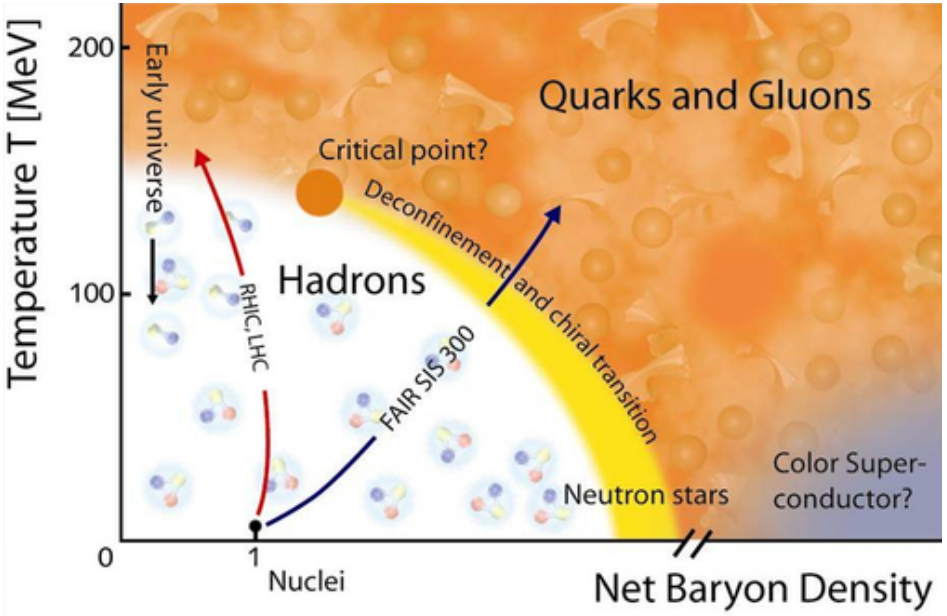
Input

- Functional equations: propagators, three-point functions

QCD phase diagram

Examples for accessible quantities:

- quark condensate (chiral symm.)
- dual quark condensate/dressed Polyakov loop (conf.)
- dual density (conf.)
- (dual) quark dressing function (chiral symm./conf.)
- Polyakov loop potential (conf.)



Input

- Functional equations: propagators, three-point functions
- Lattice: propagators at $\mu = 0$, three-point functions
- Models

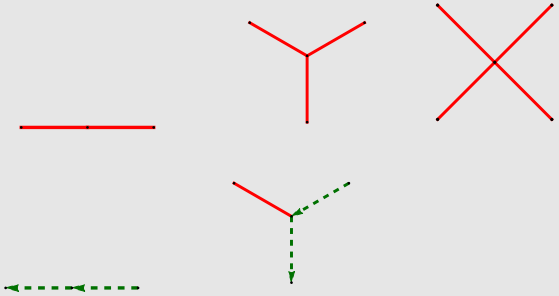
Landau gauge Yang-Mills theory

Gluonic sector of quantum chromodynamics: Yang-Mills theory

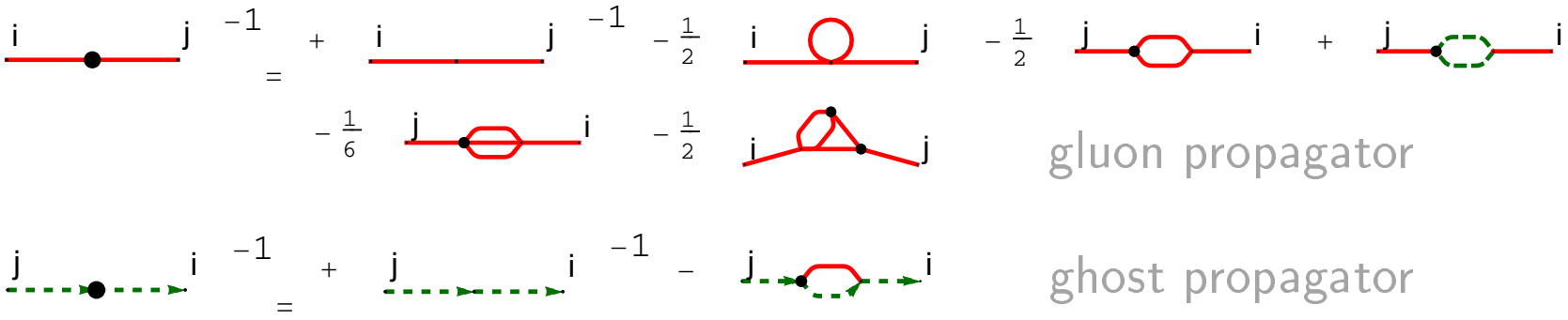
$$\mathcal{L} = \frac{1}{2} F^2 + \mathcal{L}_{gf} + \mathcal{L}_{gh}$$
$$F_{\mu\nu} = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu + i\, g\, [\boldsymbol{A}_\mu, \boldsymbol{A}_\nu]$$

Landau gauge

- simplest one for functional equations
- $\partial_\mu \boldsymbol{A}_\mu = 0$: $\quad \mathcal{L}_{gf} = \frac{1}{2\xi} (\partial_\mu \boldsymbol{A}_\mu)^2, \quad \xi \to 0$
- requires ghost fields: $\quad \mathcal{L}_{gh} = \bar{\boldsymbol{c}} \left(-\Box + g\, \boldsymbol{A} \times \right) \boldsymbol{c}$



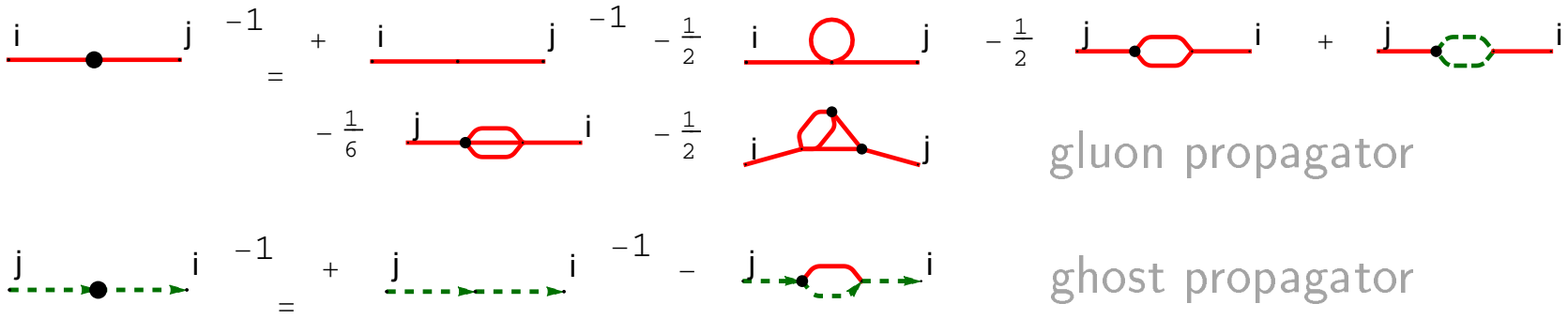
The tower of DSEs



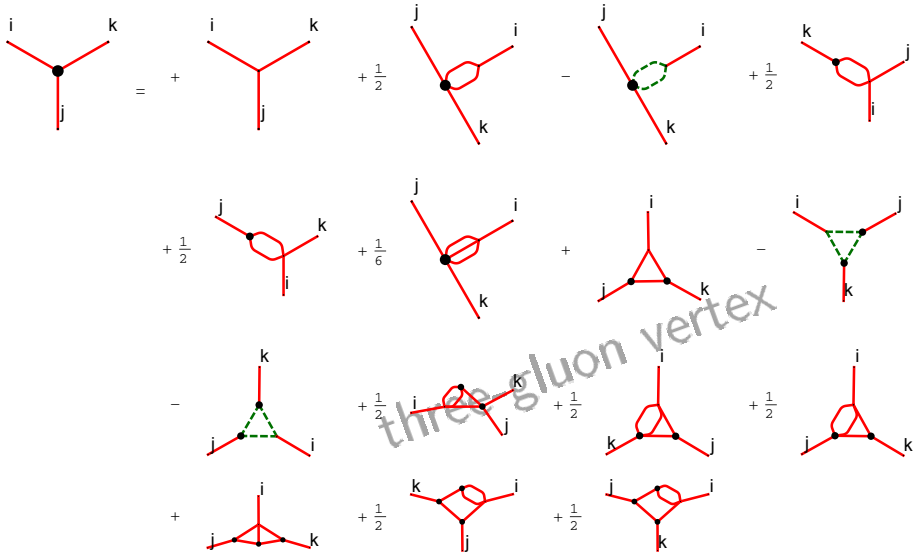
gluon propagator

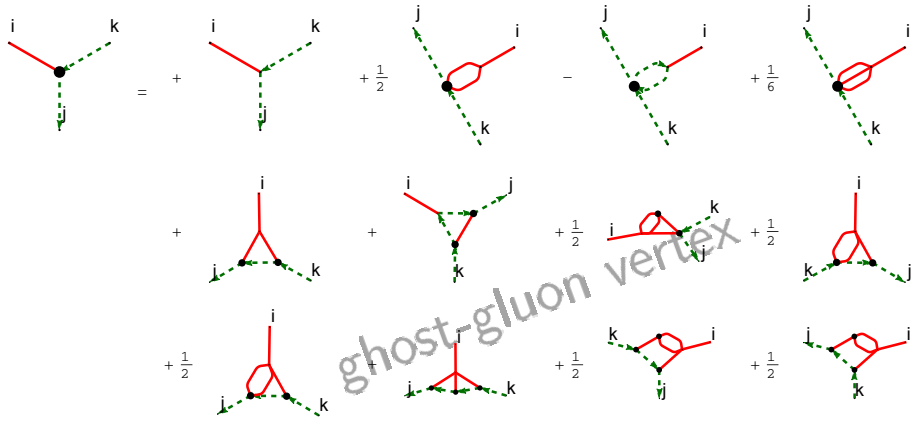
ghost propagator

The tower of DSEs



Self-consistence!





Infinitely many equations. In QCD, every n-point function depends on $(n+1)$- and possibly $(n+2)$-point functions.

Taming the equations

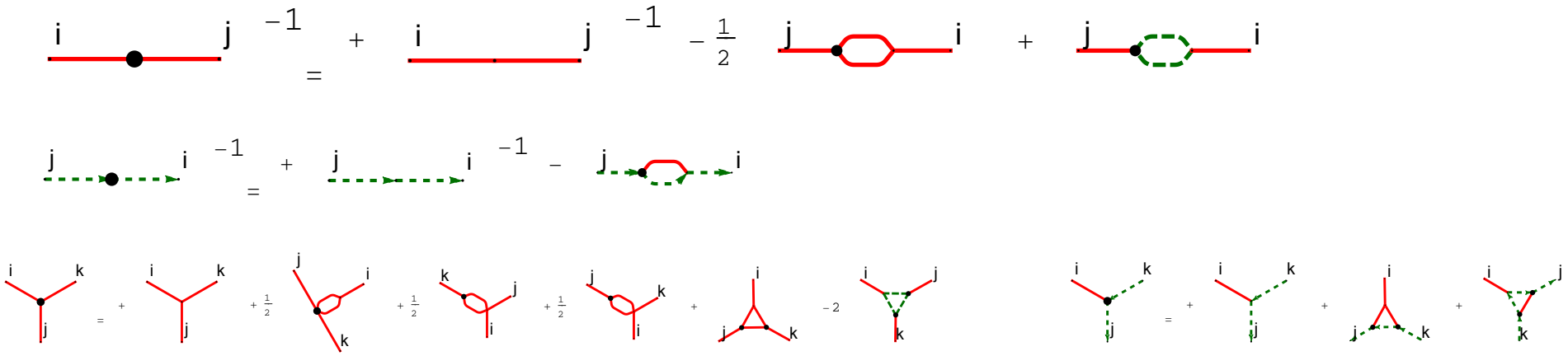
Keep most important parts!

- Drop quantities
- Model quantities

Taming the equations

Keep most important parts! The art ...

- Drop quantities
- Model quantities

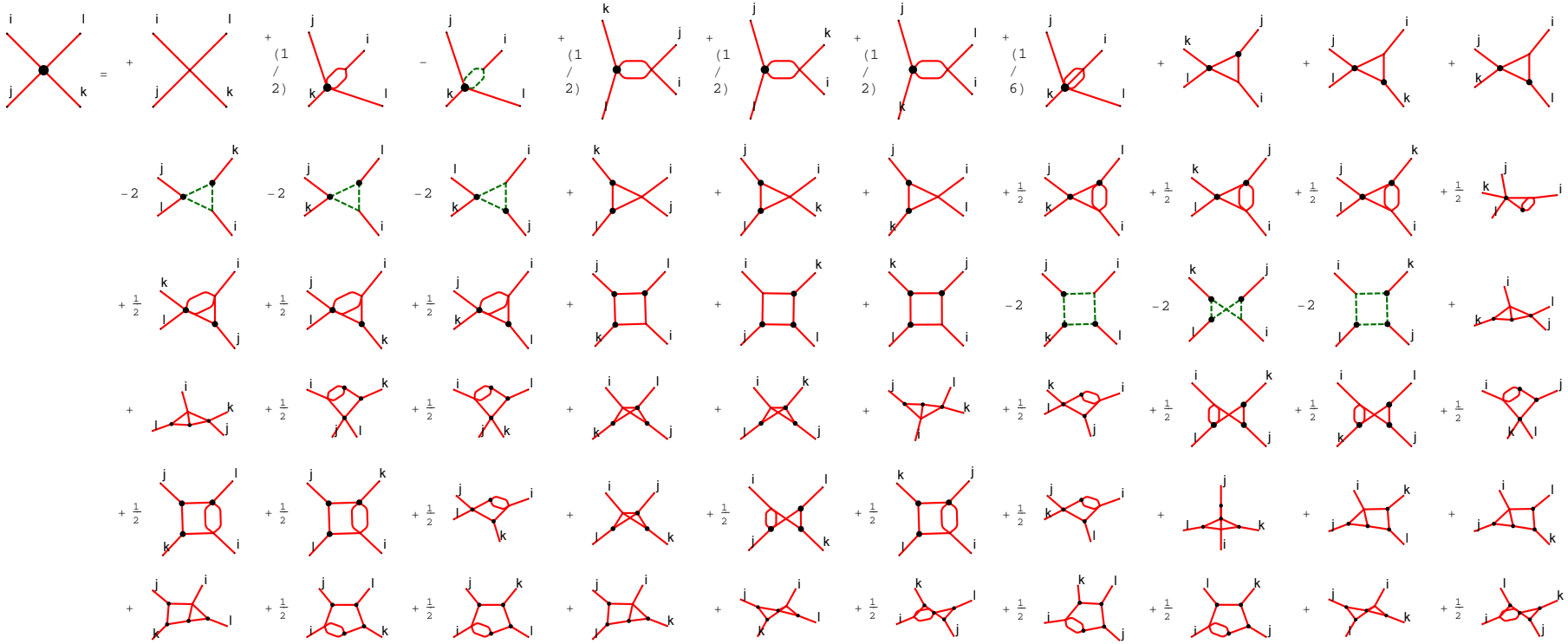


Most important parts

- UV leading (perturbation theory)
- IR leading (analytic, lattice)

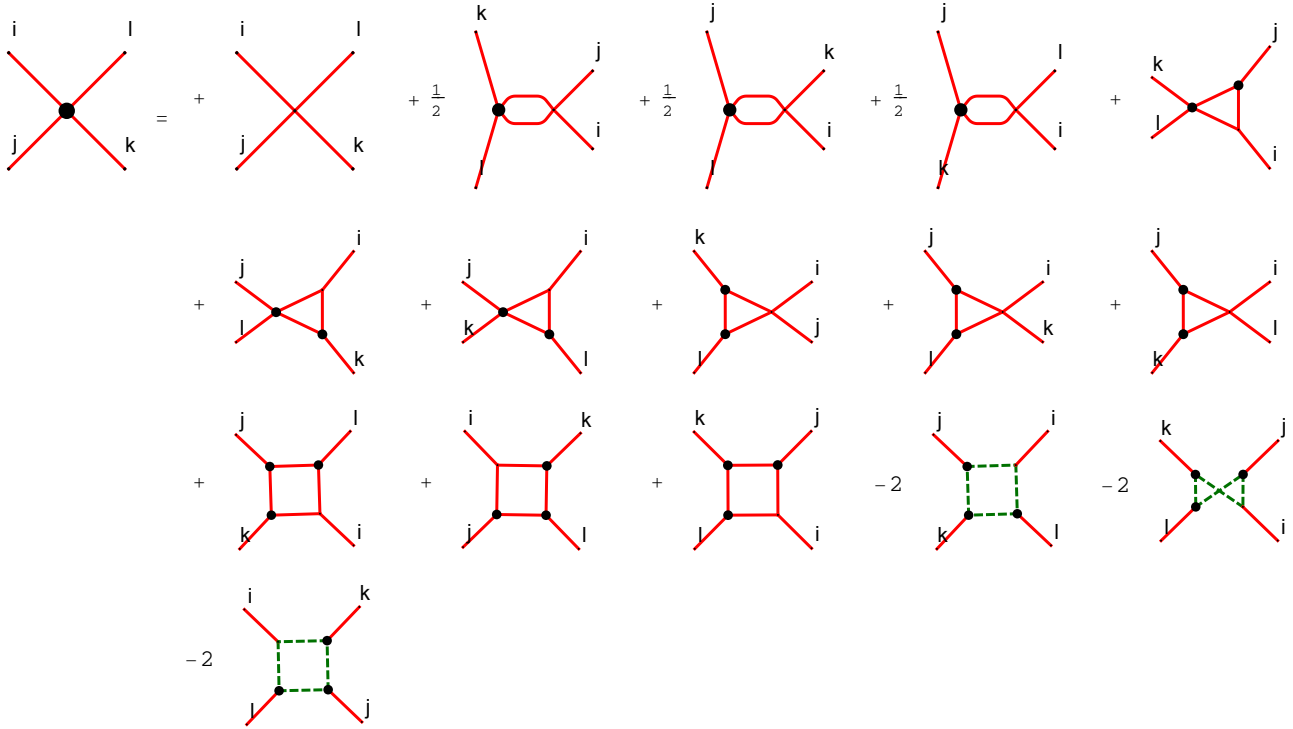
Four-gluon vertex

- 20 one-loop, 39 two-loop diagrams



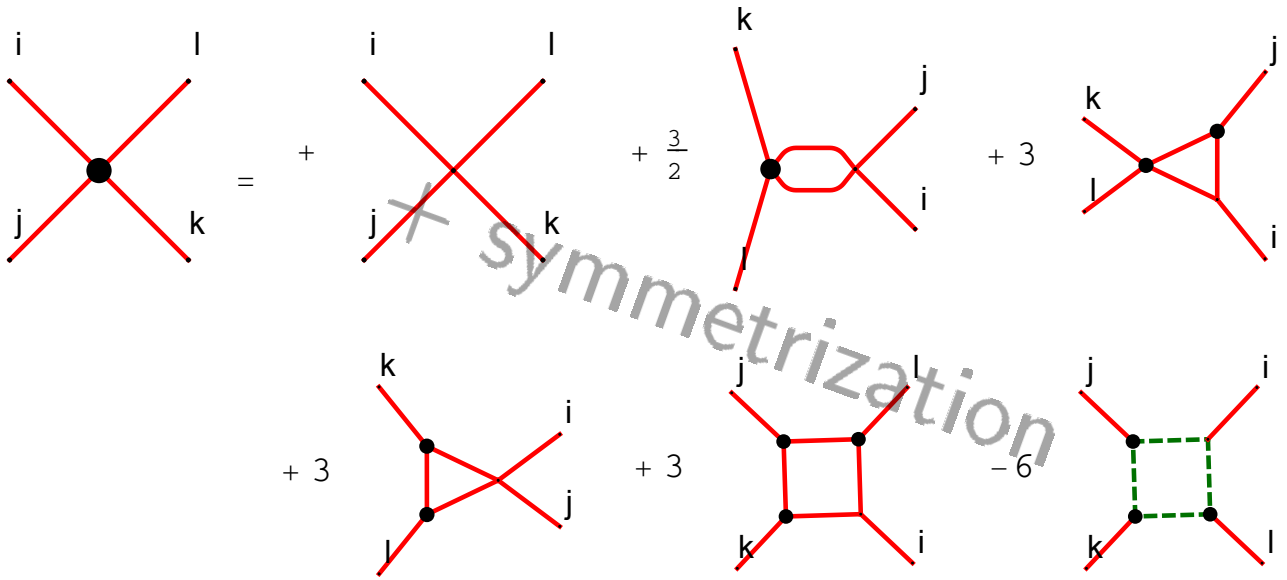
Four-gluon vertex

- 20 one-loop, 39 two-loop diagrams
- Keep UV leading diagrams $\rightarrow$ 16 diagrams



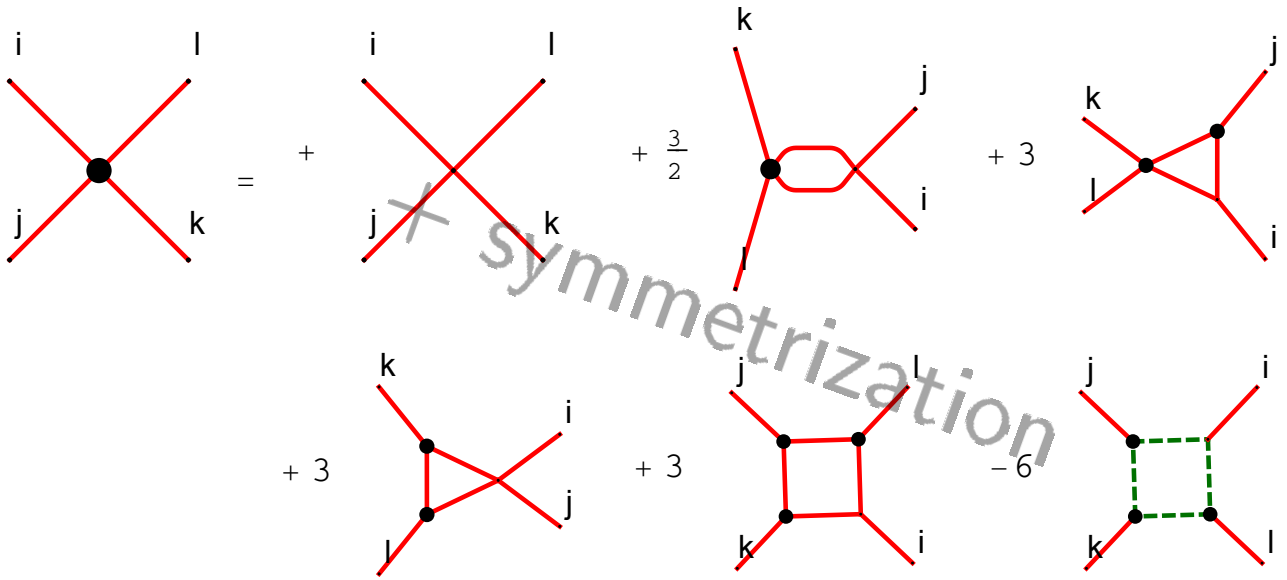
Four-gluon vertex

- 20 one-loop, 39 two-loop diagrams
- Keep UV leading diagrams → 16 diagrams
- Calculate **full** momentum dependence.
  → Access to all permutations of this diagram. → 6 diagrams



Four-gluon vertex

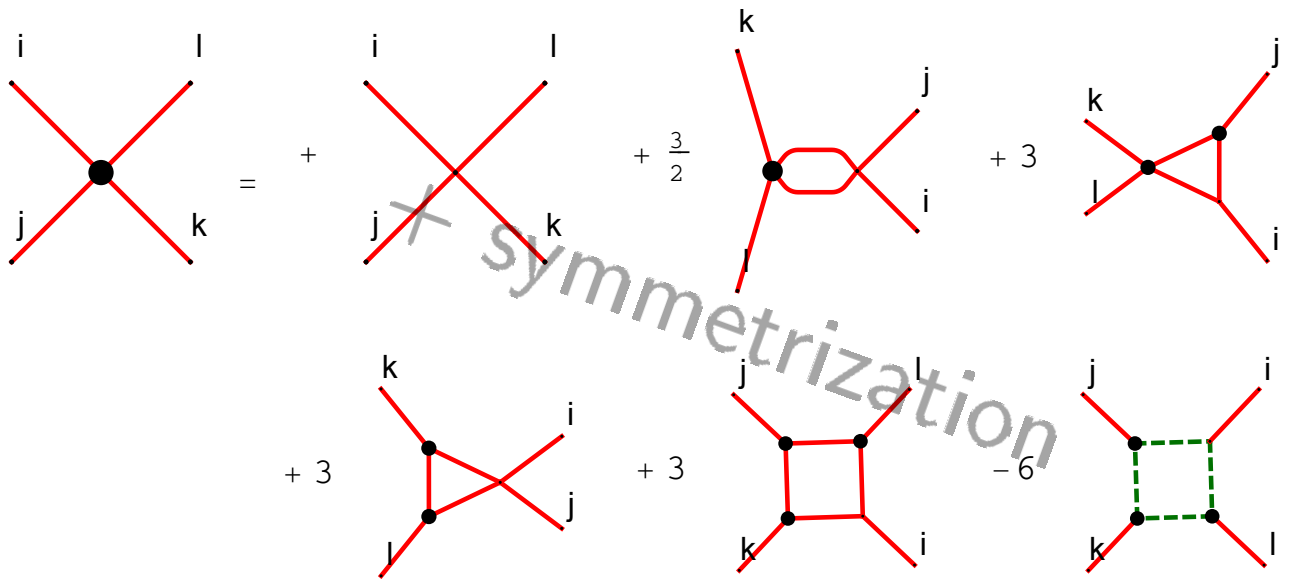
- 20 one-loop, 39 two-loop diagrams
- Keep UV leading diagrams → 16 diagrams
- Calculate **full** momentum dependence.
  → Access to all permutations of this diagram. → 6 diagrams



No model dependence! → 'Truncation closes.'

Four-gluon vertex

- 20 one-loop, 39 two-loop diagrams
- Keep UV leading diagrams → 16 diagrams
- Calculate **full** momentum dependence.
  → Access to all permutations of this diagram. → 6 diagrams



- 6 external variables
- 4 integration variables

No model dependence! → 'Truncation closes.'

Truncation recap

Principle

- Drop or model unknown quantities ...
- ... but capture the important part.

Truncation recap

Principle

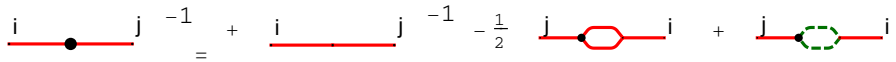
- Drop or model unknown quantities ...
- ... but capture the important part.

Yan-Mills theory:

- Complete at leading UV order:
  primitively divergent quantities only. $\rightarrow$ 5 coupled DSEs
- Truncation closes.

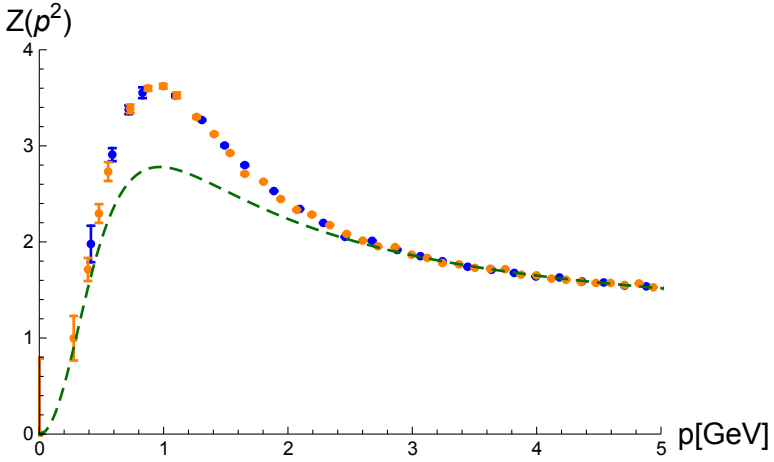
In the following consider subparts of this system.

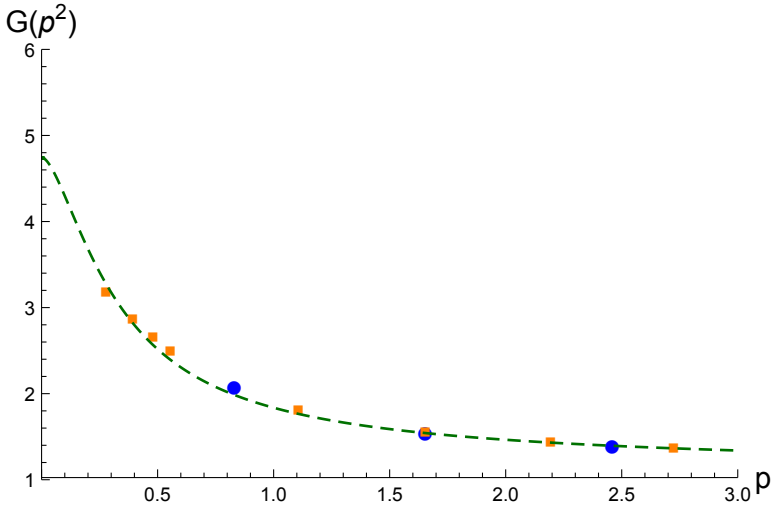
Propagators I



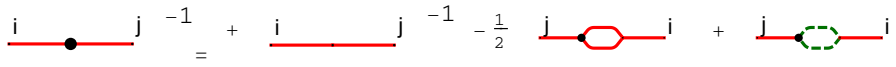
Long-time standard truncation

- No four-gluon vertex
- Ghost-gluon vertex: bare
- Three-gluon vertex: model



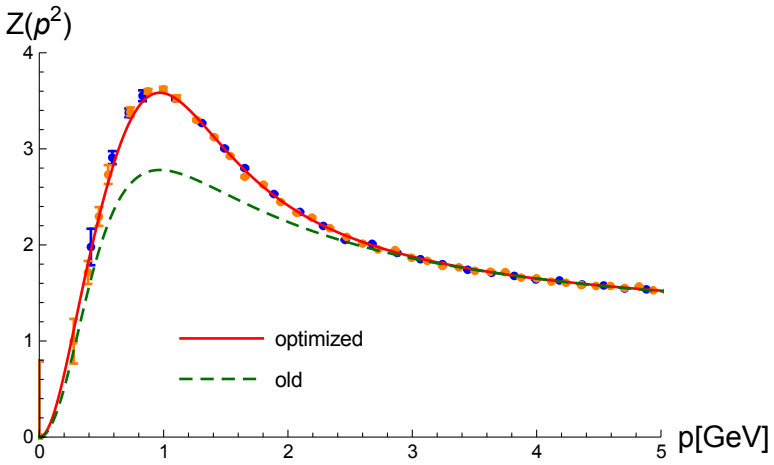


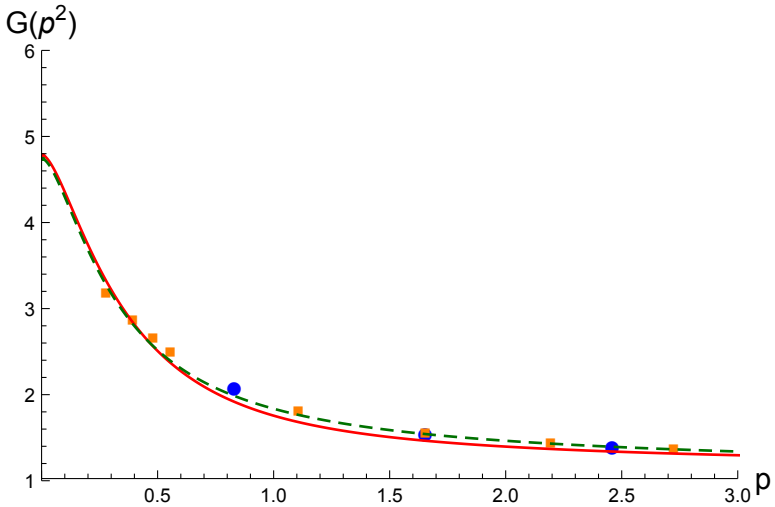
Propagators I



Long-time standard truncation

- No four-gluon vertex
- Ghost-gluon vertex: bare → dressed (dynamic)
- Three-gluon vertex: model → optimized model



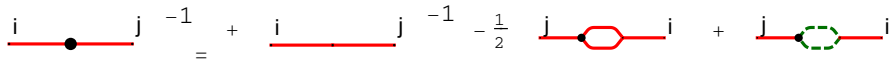


[MQH, von Smekal '12; lattice; Sternbeck '06]

→ Role of three-gluon vertex?

→ Use as input in other calculations.

Propagators I

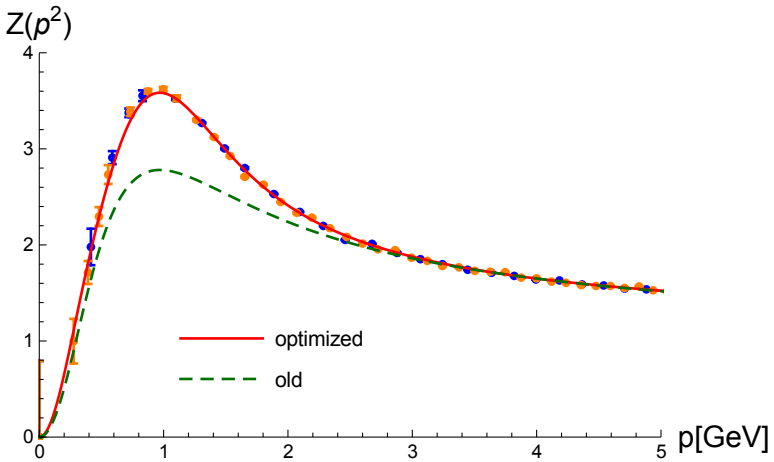


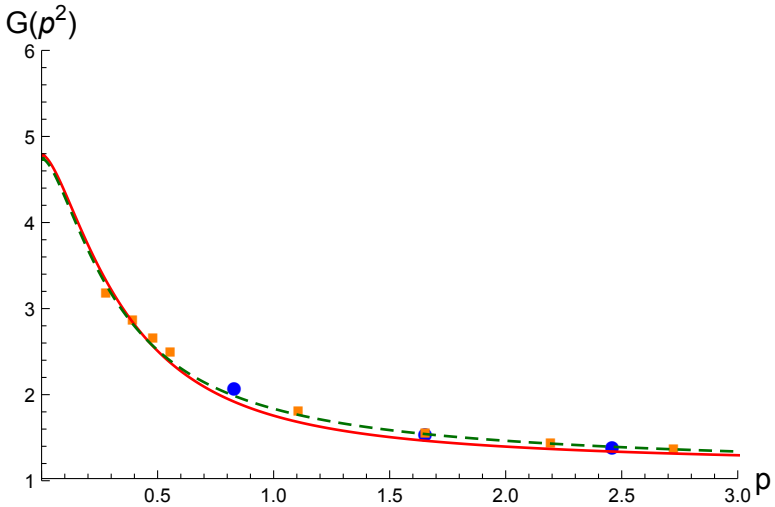
Long-time standard truncation

- No four-gluon vertex
- Ghost-gluon vertex: bare → dressed (dynamic)
- Three-gluon vertex: model → optimized model

Missing strength in mid-momentum regime:

- neglected diagrams?
- vertices?





[MQH, von Smekal '12; lattice; Sternbeck '06]

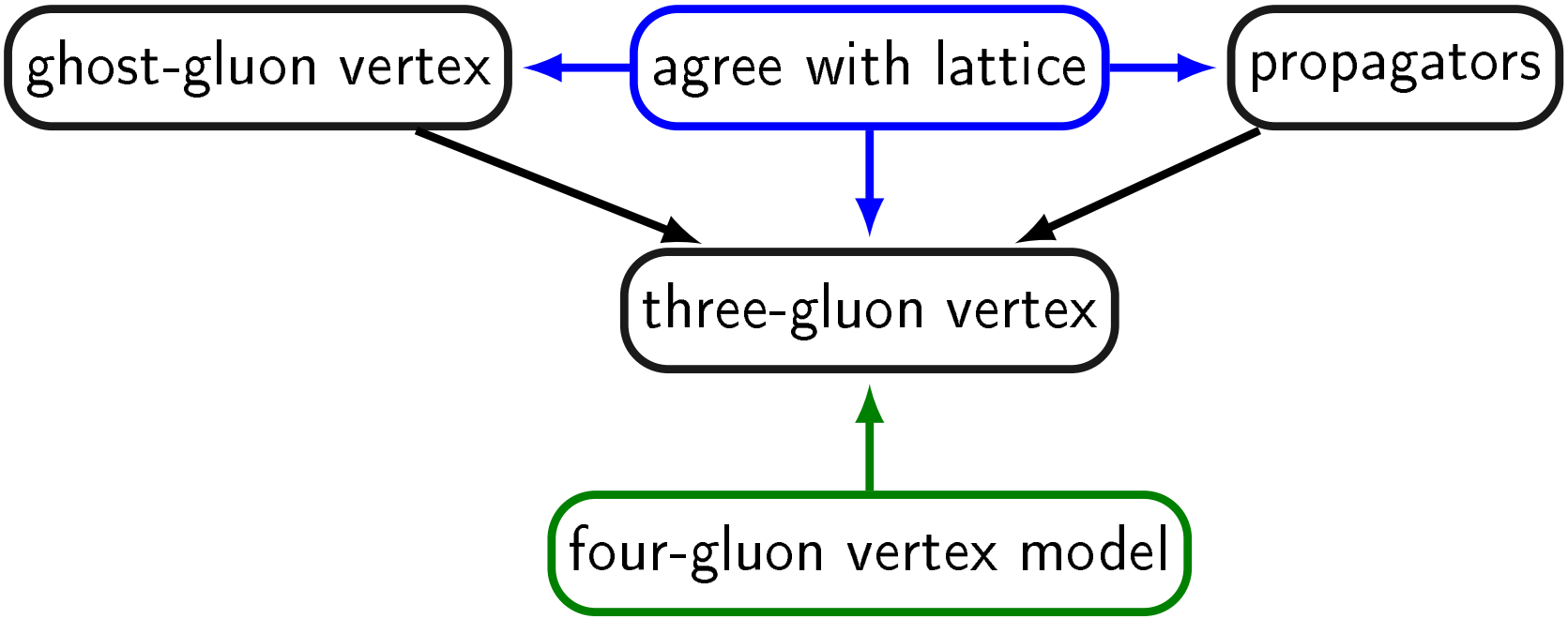
→ Role of three-gluon vertex?

→ Use as input in other calculations.

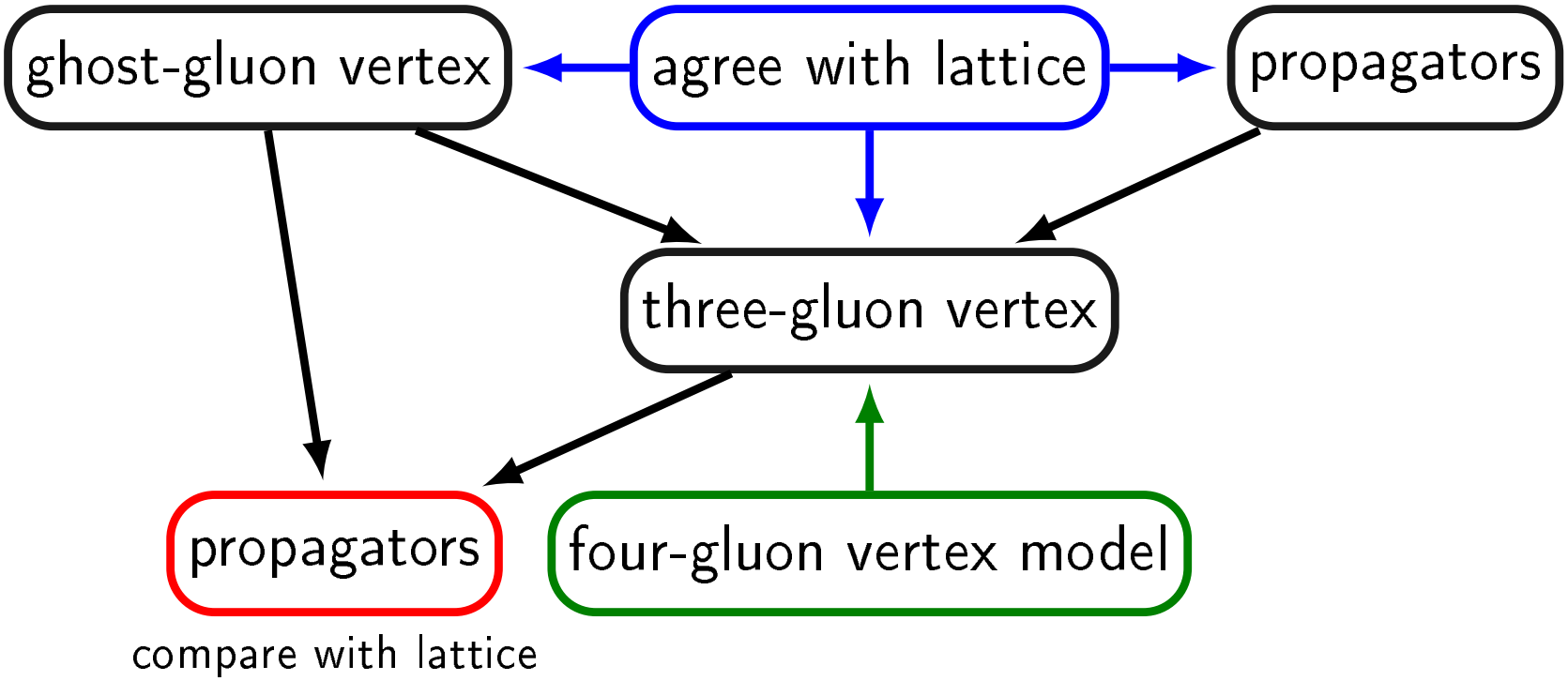
Strategy

ghost-gluon vertex ← agree with lattice → propagators

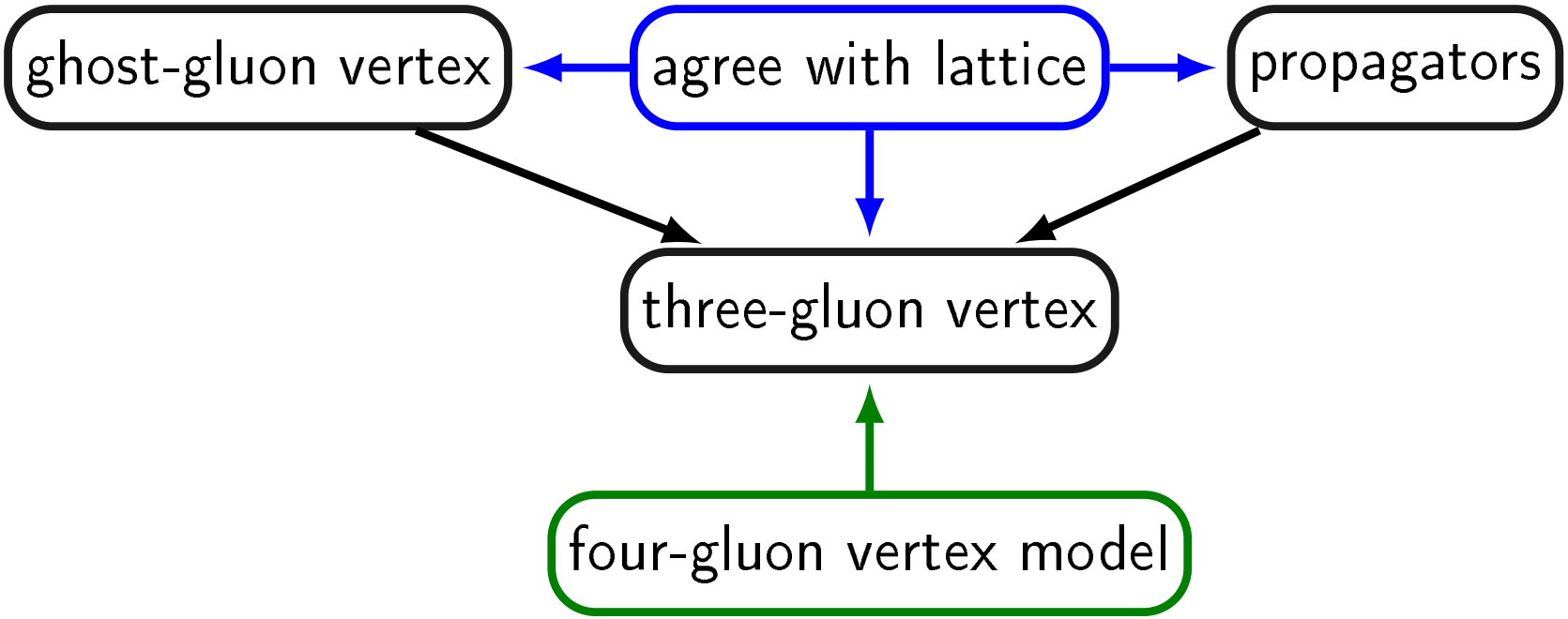
Strategy


ghost-gluon vertex
agree with lattice
propagators
three-gluon vertex
four-gluon vertex model

Strategy


ghost-gluon vertex
agree with lattice
propagators
three-gluon vertex
propagators
four-gluon vertex model
compare with lattice

Strategy


ghost-gluon vertex
agree with lattice
propagators
three-gluon vertex
four-gluon vertex model

The three-gluon vertex I

Four-gluon vertex model (decoupling):

$$D^{A^4}(p,q,r,s) = (a \tanh(b/\bar{p}^2) + 1)\, D^{A^4}_{RG}(p,q,r,s)$$

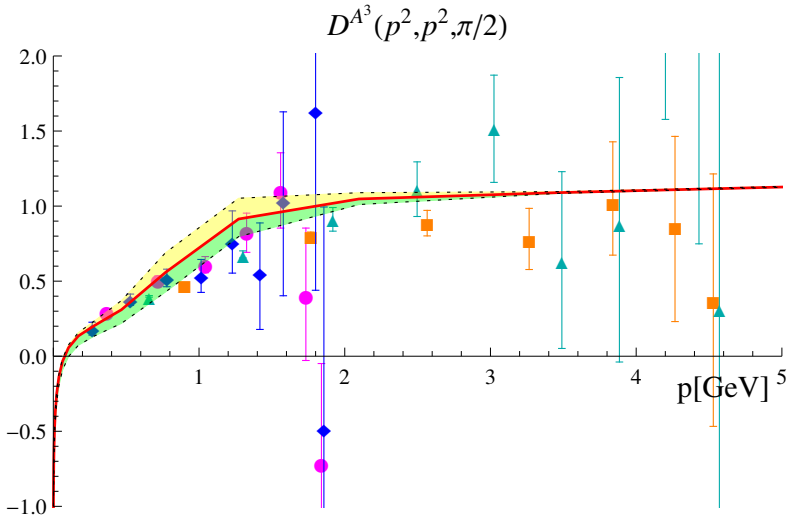
$\rightarrow$ Test model dependence by varying a and b.

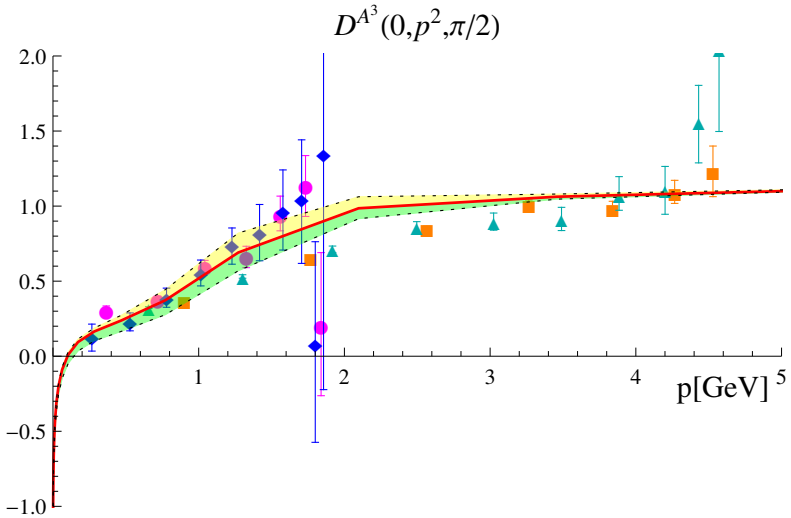
The three-gluon vertex I

Four-gluon vertex model (decoupling):

$$D^{A^4}(p,q,r,s) = (a\tanh(b/\bar{p}^2)+1)\, D^{A^4}_{RG}(p,q,r,s)$$

→ Test model dependence by varying a and b.





[Blum, MQH, Mitter, von Smekal '14; lattice: Cucchieri, Maas, Mendes '08]

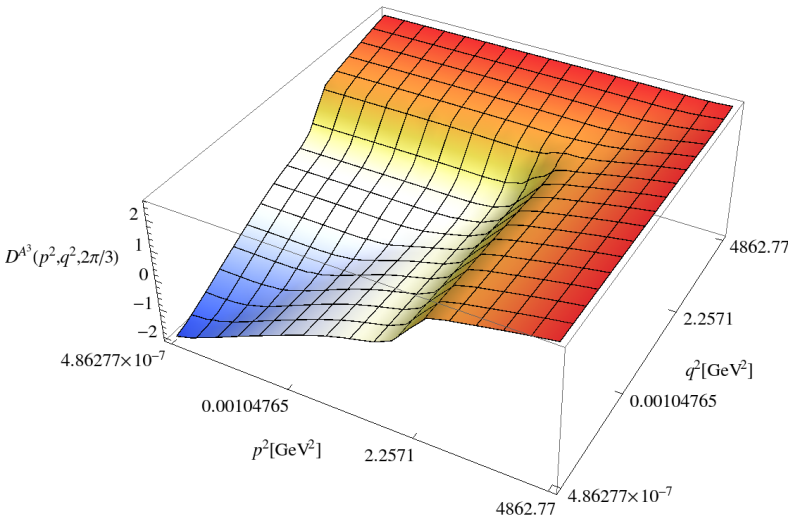
→ Truncation reliable. Neglected terms, including two-loop, suppressed.

See also results by [Eichmann, Williams, Alkofer, Vujinovic '14], esp. other dressings, and [Peláez, Tissier, Wschebor '13].

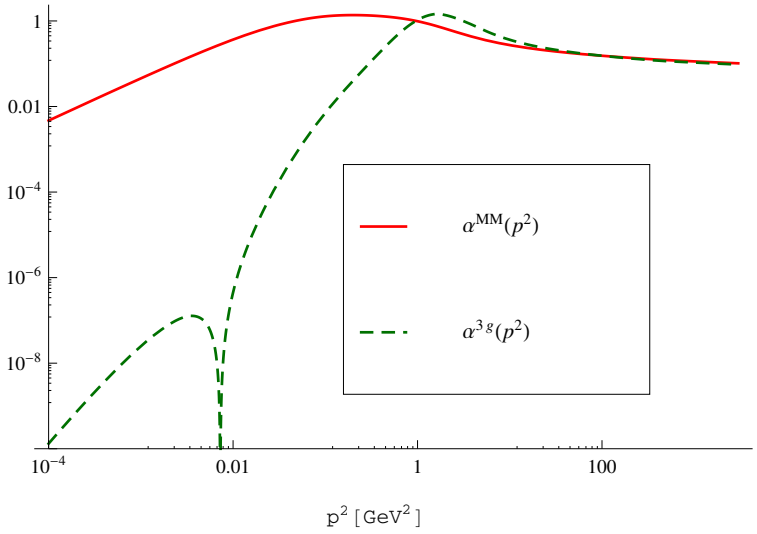
The three-gluon vertex II

$$\Gamma^{AAA,abc}_{\mu\nu\rho}(p,q,k) := i\,g\,f^{abc}\,D^{AAA}(p^2,q^2,\cos\theta)\Gamma^{AAA,(0)}_{\mu\nu\rho}(p,q,k)$$

Fixed angle:

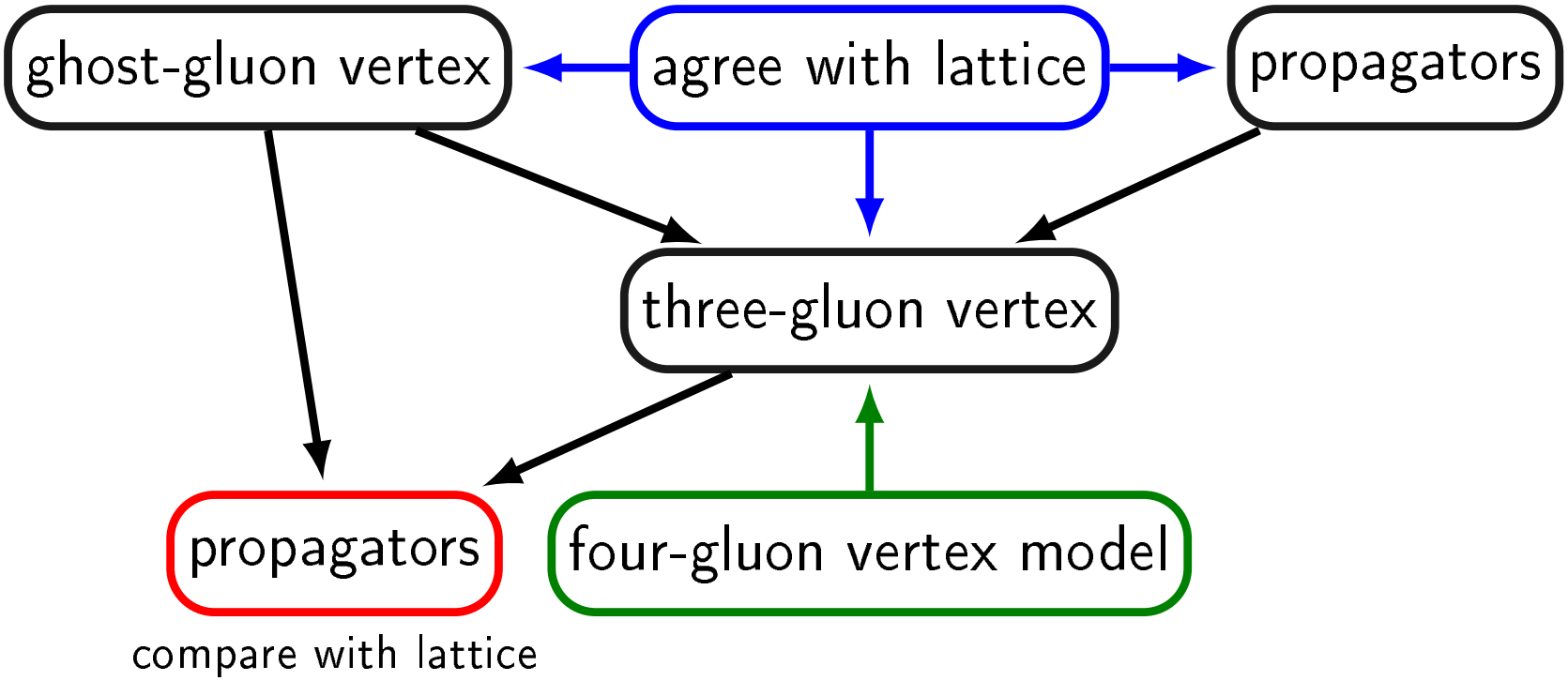


Couplings:

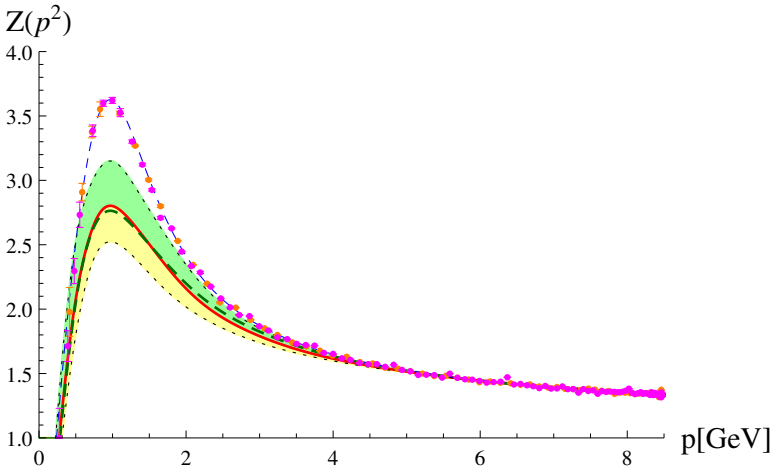


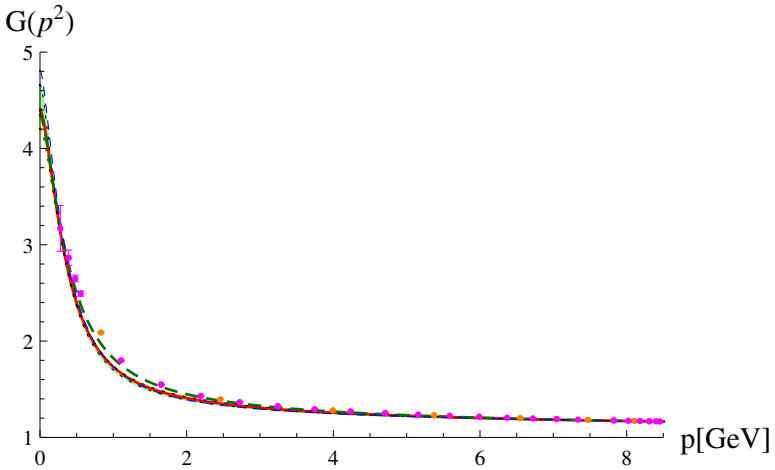
- Bose symmetry visible (enforced by hand).
- Zero crossing (in this tensor), also found by [Peláez, Tissier, Wschebor '13; Aguilar, Binosi, Ibáñez, Papavassiliou '13; Eichmann, Williams, Alkofer, Vujinovic '14] and on lattice in $d = 2, 3$.

Strategy


ghost-gluon vertex
agree with lattice
propagators
three-gluon vertex
propagators
four-gluon vertex model
compare with lattice

Propagators II: Limits of one-loop truncation





[Blum, MQH, Mitter, von Smekal '13]

- Ghost almost unaffected.
- Gap in midmomentum regime must be due to missing two-loop diagrams!

NB: Employed projection of three-gluon vertex is the same as in gluon loop of gluon propagator DSE! $\rightarrow$ Error from neglected tensors negligible.

Explicit two-loop studies [Bloch '03; Mader, Alkofer '12; Meyers, Swanson '14]:
squint $\gg$ sunset diagram

Four-gluon vertex

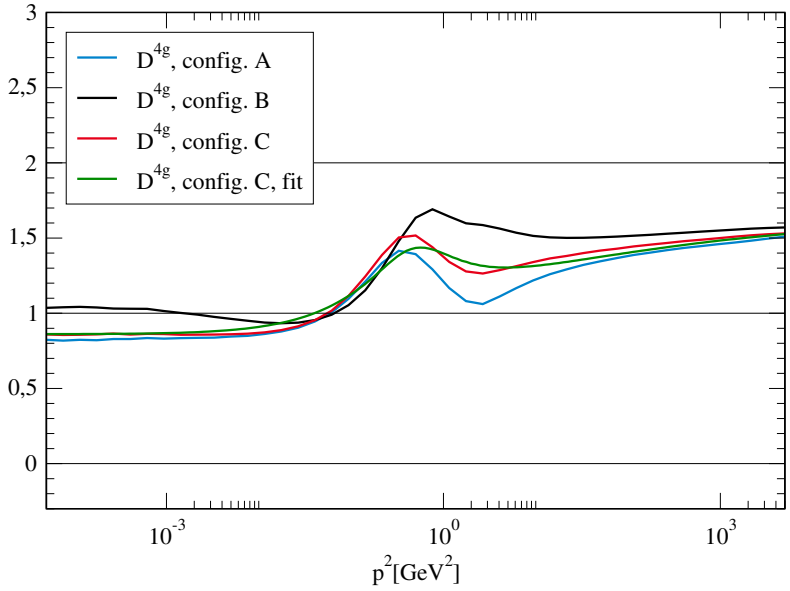
- 6 external variables
- 4 integration variables

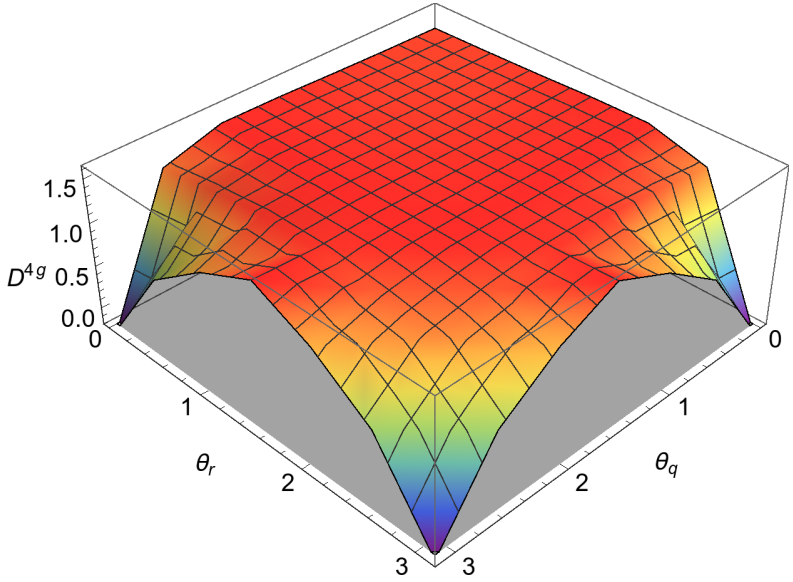
Cf. propagator: 1 ext., 2 int.
2 propagators $\rightarrow$ laptop
four-gluon vertex $\rightarrow > 100$ cores on cluster

Four-gluon vertex

- 6 external variables
- 4 integration variables

Cf. propagator: 1 ext., 2 int.
2 propagators → laptop
four-gluon vertex → > 100 cores on cluster





[Cyrol, MQH, von Smekal '14]

2-parameter fit:

$$D^{4g,\,\mathrm{dec}}_{\mathrm{model}}(p,\,q,\,r,\,s) = \left(a \tanh\left(b/\bar{p}^2\right) + 1\right) D^{4g}_{\mathrm{RG}}(p,\,q,\,r,\,s)$$

Beyond Landau gauge: Coulomb gauge

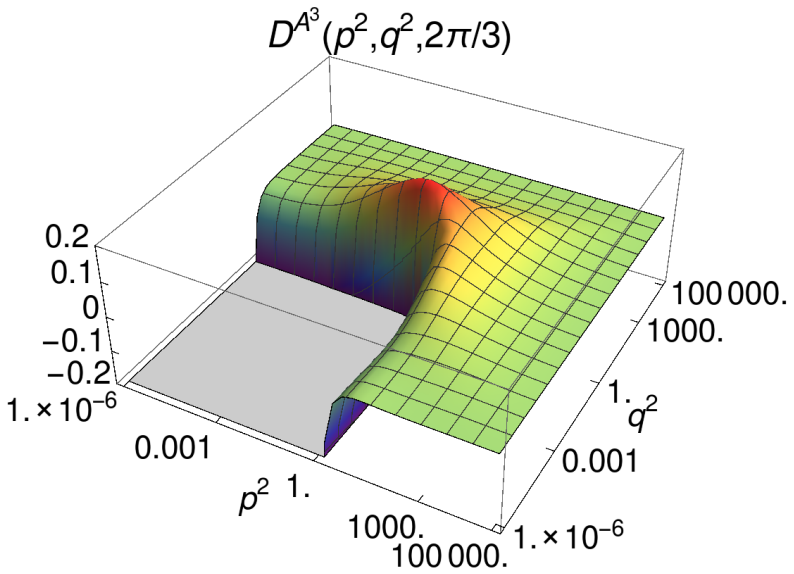
Why the Landau gauge is convenient

- Minimum number of terms in DSEs.
- Transversality $\rightarrow$ longitudinal part decouples.
- Historically ghost-gluon vertex provided the entry point (special here).

Beyond Landau gauge: Coulomb gauge

Why the Landau gauge is convenient

- Minimum number of terms in DSEs.
- Transversality → longitudinal part decouples.
- Historically ghost-gluon vertex provided the entry point (special here).



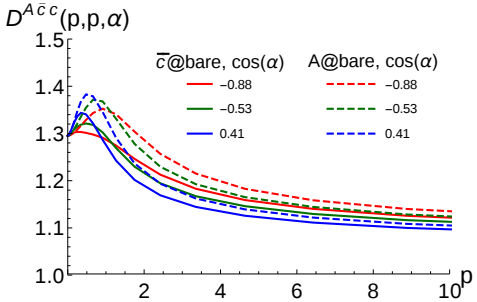
[MQH, Campagnari, Reinhardt '14]

Three-gluon vertex:

- Zero crossing
- IR divergent like p^{-3}

Ghost-gluon vertex:

- Different truncations quite similar



Summary

Highlighted prospects:

- Description of hadron properties from first principles.
- QCD phase diagram: explore non-zero density.

Summary

Highlighted prospects:

- Description of hadron properties from first principles.
- QCD phase diagram: explore non-zero density.

Technical challenges!

Summary

Highlighted prospects:

- Description of hadron properties from first principles.
- QCD phase diagram: explore non-zero density.

Technical challenges!

Truncation:

- 2-, 3- and 4-point functions calculated
- Truncation effects understood (after more than 30 years!)
- Two-loop terms important in 2- but not in higher n-functions
- System of DSEs closes with this truncation

$\rightarrow$ self-contained, quantitative description.

Summary

Highlighted prospects:

- Description of hadron properties from first principles.
- QCD phase diagram: explore non-zero density.

Technical challenges!

Truncation:

- 2-, 3- and 4-point functions calculated
- Truncation effects understood (after more than 30 years!)
- Two-loop terms important in 2- but not in higher n-functions
- System of DSEs closes with this truncation

$\rightarrow$ self-contained, quantitative description.

Thank you for your attention.