Using Dyson-Schwinger equations to investigate Yang-Mills theory in the infrared: scaling solutions, power counting and infrared exponents

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FWF Der Wissenschaftsfonds.

SIC!QFT UNI GRAZ

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Phenomenology of the strong interaction

Particles under the influence of the strong force: hadrons, e. g. π, K, η, proton, neutron, Λ, Σ, ...

- High energy experiments: point-like particles inside the hadrons (quarks).
- Quarks only exist in bound states, never as free particles (confinement).
- Mediator of the strong force: gluons (also confined).
- Theory: Quantum Chromodynamics (QCD).
- At high energies QCD is asymptotically free, i. e. the coupling gets small and we can "observe" quarks (Nobel prize 2004).
- At lower energies non-perturbative methods are needed.

This talk

In this talk I will focus on the low energy behavior of Yang-Mills theory (gluonic part of QCD).

Confinement of quarks and gluons

- Confinement is a long-range $\leftrightarrow$ IR phenomenon: We do not see individual $\sim$ infinitely separated quarks or gluons.
- One expects that the property of being confined is encoded in the particles' propagators.
- Different confinement criteria for the propagators:
  - Positivity violations: negative norm contributions $\longrightarrow$ not a particle of the physical state space
  - Gribov-Zwanziger (Landau gauge, Coulomb gauge): IR suppression of the gluon propagator $\longrightarrow$ no long-distance propagation
  - Kugo-Ojima: quartet mechanism, e. g. Gupta-Bleuler formalism in QED: timelike and longitudinal photon cancel each other

Functional methods employ

correlation functions/Green fcts./n-point fcts./propagators and vertices.

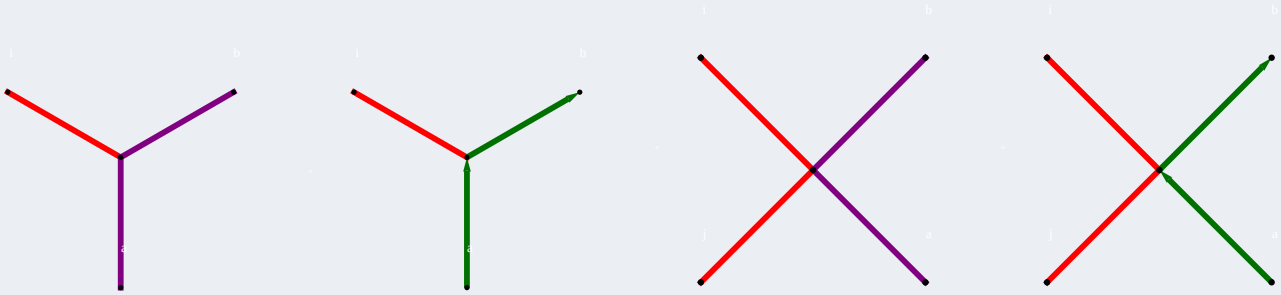
The equations of motion of these are the Dyson-Schwinger equations.

Propagators and vertices

The theory is encoded in the Green functions: "building blocks" for functional equations.
They describe propagation and interactions of fields.

Graphical notation

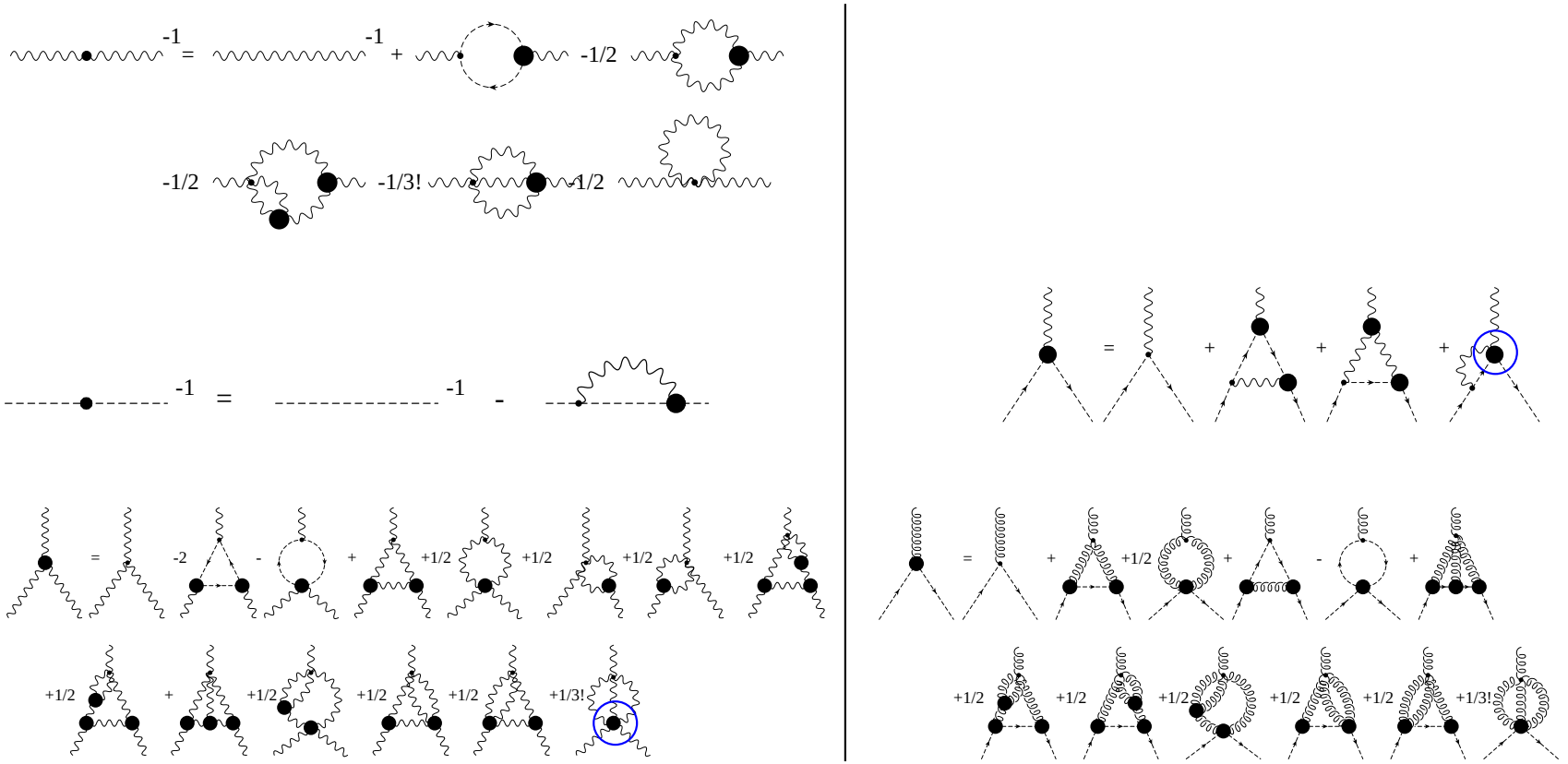
Propagators: 

Vertices: 

The propagators and interactions are given by the Lagrangian of the theory.
Shorthand notation: propagator of field A is AA, quartic interaction is $AAAA$ etc.

The tower of DSEs

DSE describe non-perturbatively how particles propagate and interact.



n-point functions couple to n-point, (n+1)- and (n+2)-point functions.

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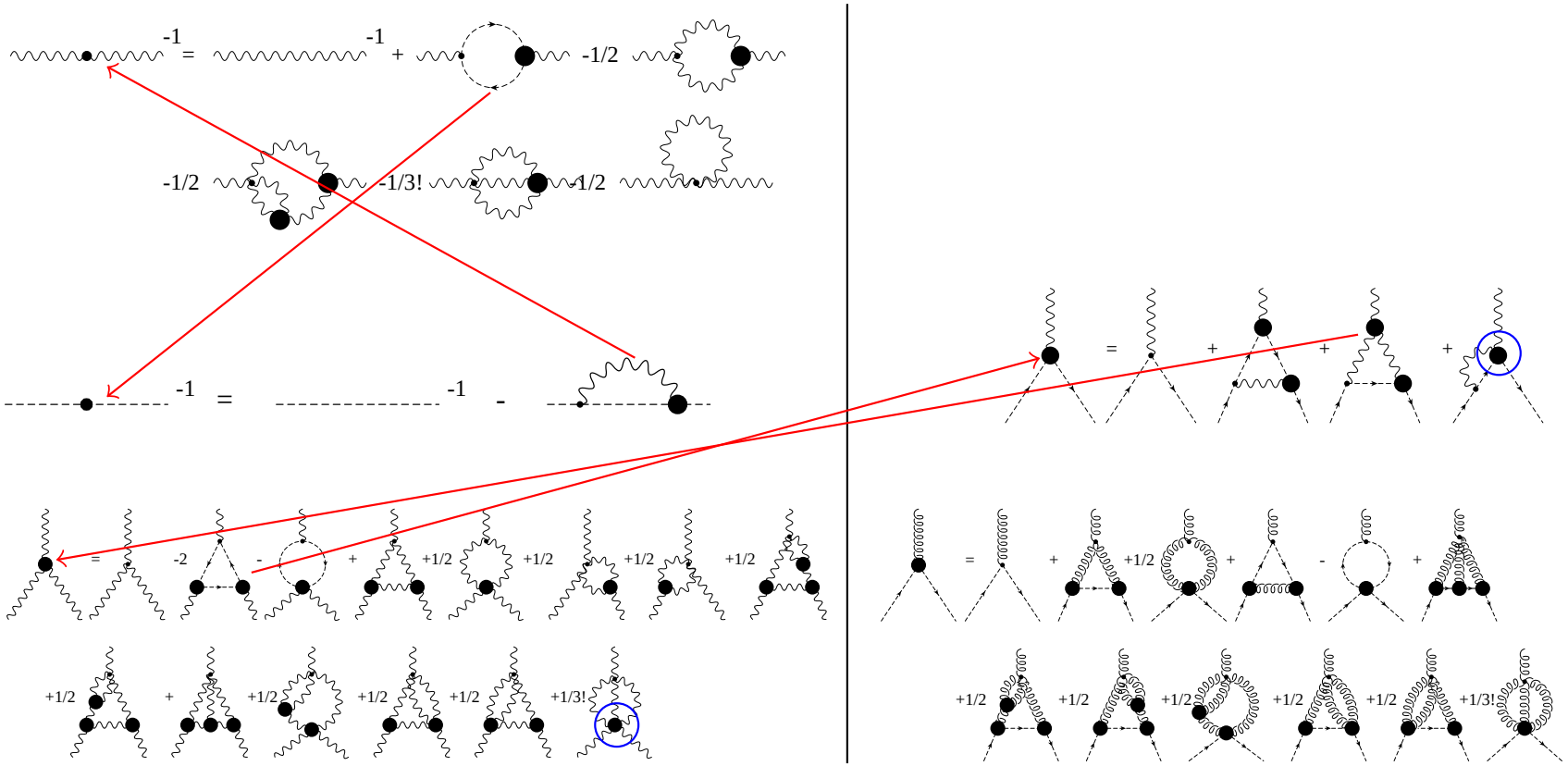

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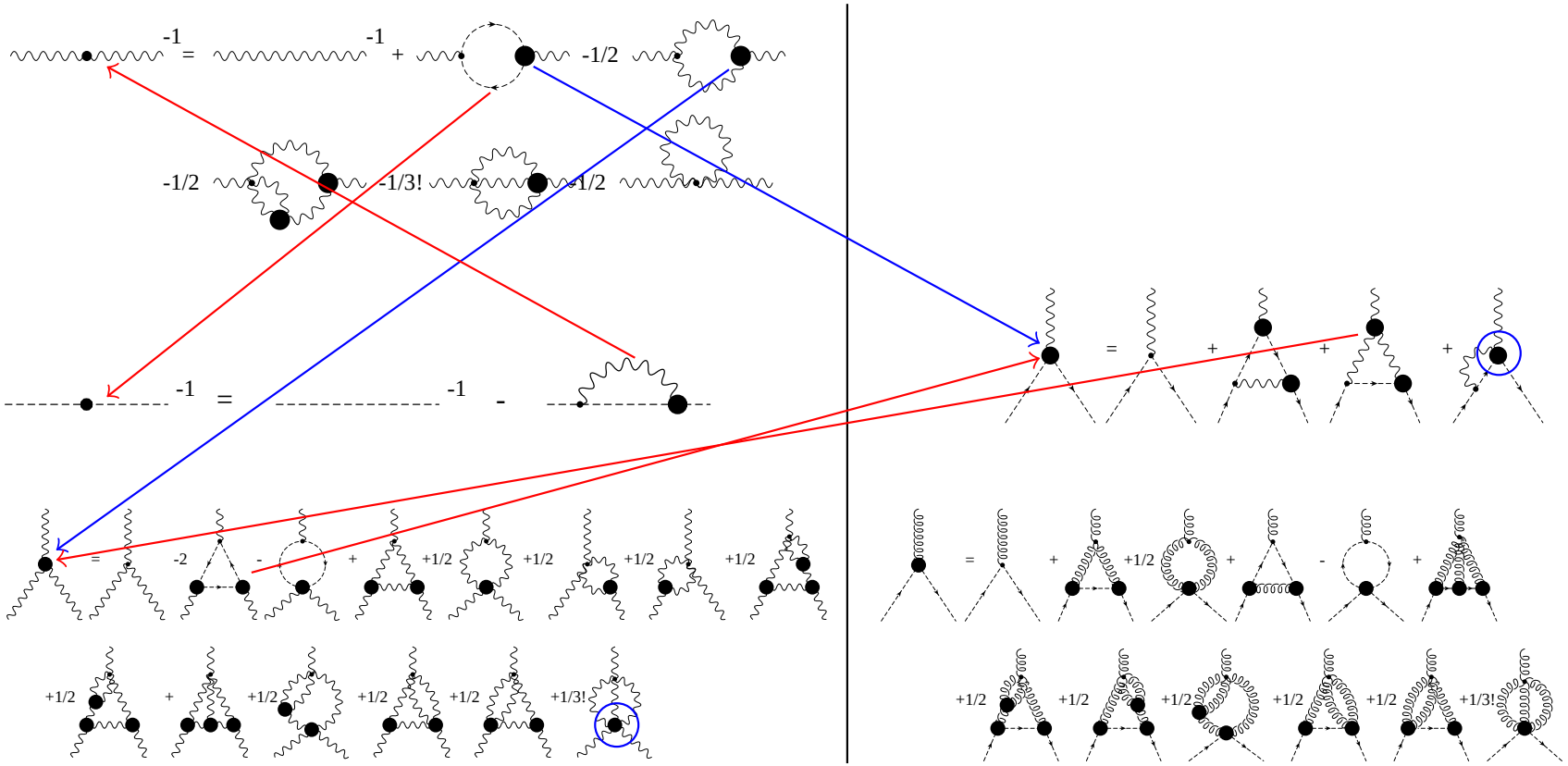

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Deriving Dyson-Schwinger equations

Starting from the translation invariance of the path integral,

$$\frac{\delta}{\delta\phi} Z[J] = \int [D\phi] \left(J - \frac{\delta S}{\delta\phi} \right) e^{-S+J\,\Phi} = 0,$$

the DSEs for all Green functions (full, connected, 1PI) can be derived by further differentiations.
After the first derivative is done, the procedure is iterative. Doing it by hand becomes tedious.

⇒ *DoDSE* [Alkofer, M.Q.H., Schwenzer, CPC 180]

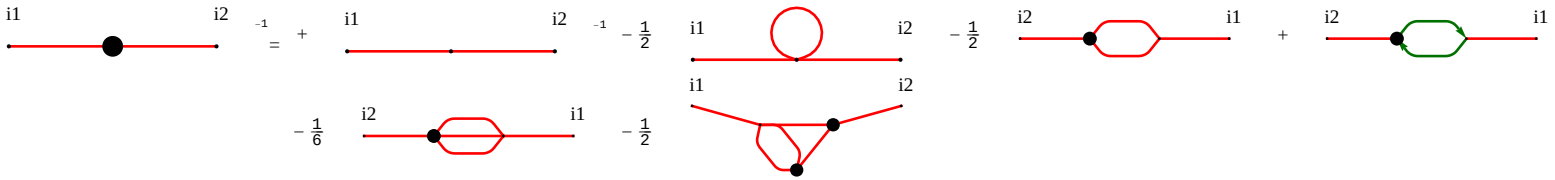
Given a structure of interactions, the DSEs are derived symbolically using *Mathematica*.

Example (Landau gauge): only one things is needed as input

- interactions in Lagrangian AA, AAA, AAAA, cc, Acc
- Which DSE do I want?

Landau Gauge: Propagators

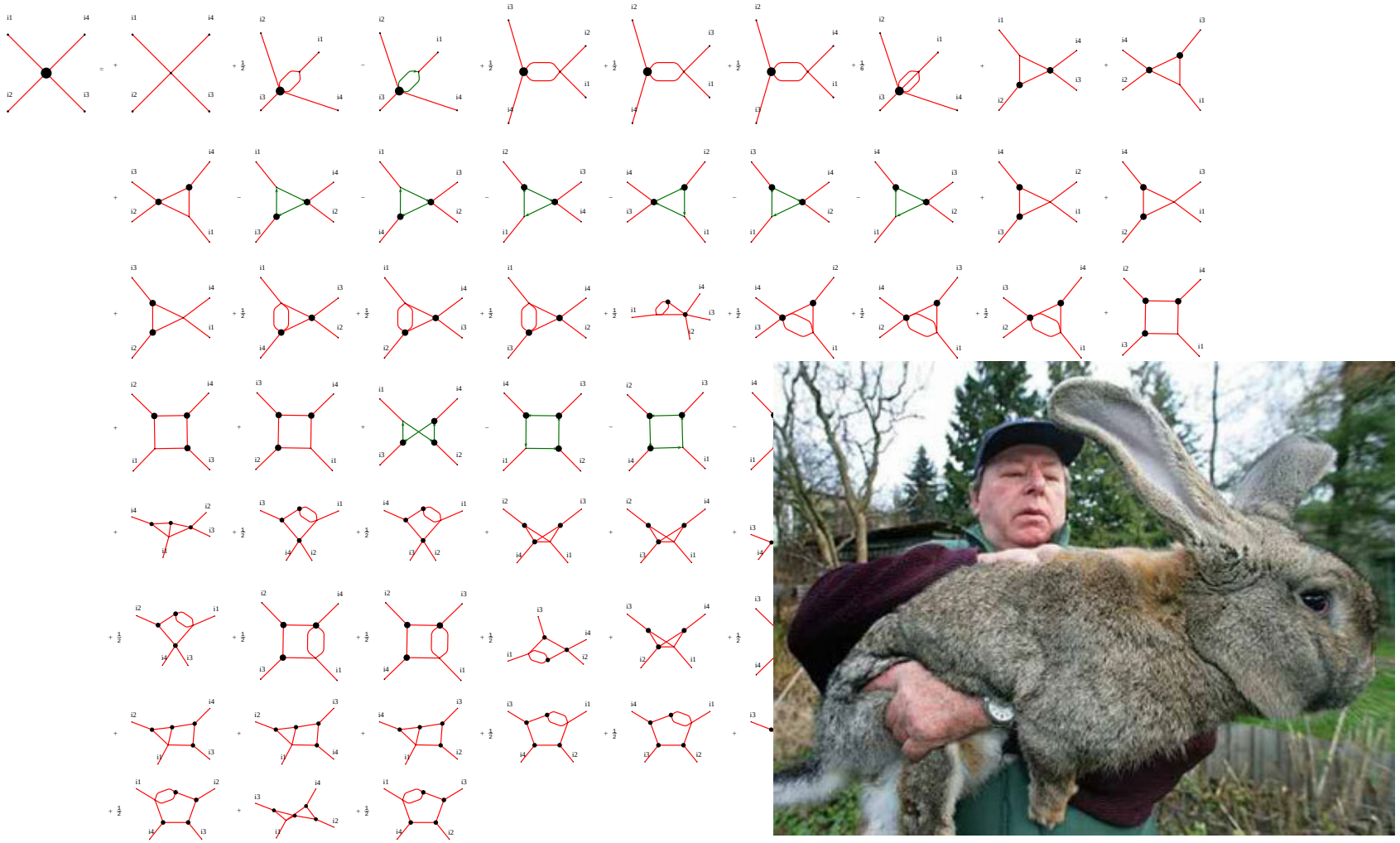
Gluon propagator:



Ghost propagator:

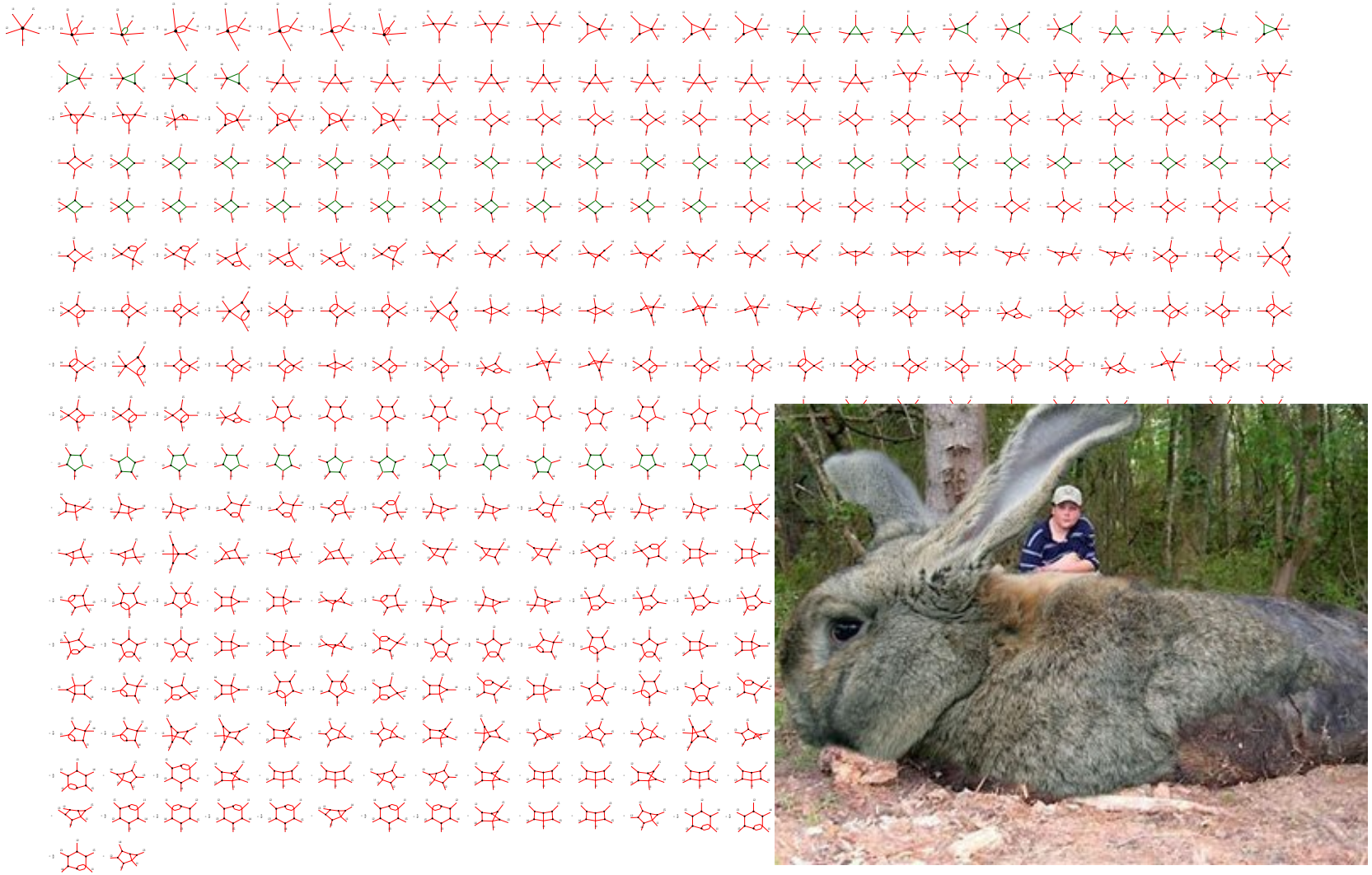
Landau Gauge: Four-Gluon Vertex

66 terms



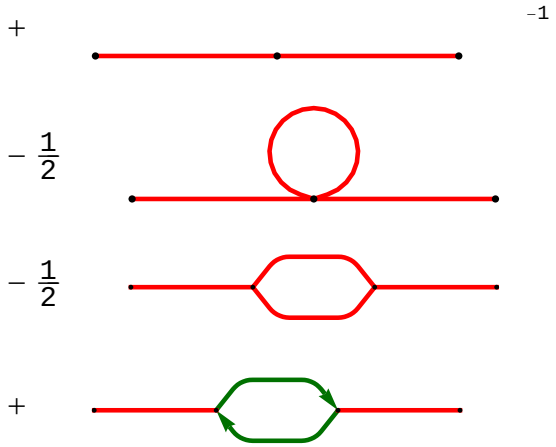
Landau Gauge: Five-Gluon Vertex

434 terms



DoDSE upgrade: *Symb2Alg*

Symbolic expressions are fine for some tasks, e. g. infrared analysis, but also algebraic expressions are needed!



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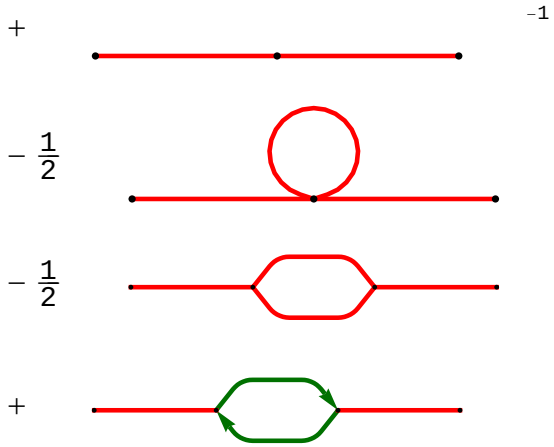

$$\frac{\left(g^{\mu 1\,\nu 1}\,\mathrm{p1}^2 - \mathrm{p1}^{\mu 1}\,\mathrm{p1}^{\nu 1}\right)\,\delta_{\mathrm{a1\,b1}}}{\mathrm{p1}^{2^2}}$$

$$-\frac{C_A\,g^2\,\left(\mathrm{q1}^{\mu 1}\,\mathrm{q1}^{\nu 1} + 2\,g^{\mu 1\,\nu 1}\,\mathrm{q1}^2\right)\,\delta_{\mathrm{a1\,b1}}}{\mathrm{q1}^{2^2}}$$

$$-\frac{C_A\,g^2\,(\ll 1 \gg)\,\delta_{\mathrm{a1\,b1}}}{2\,\ll 2 \gg^{2^2}\,\left(\mathrm{p1}^2 + 2\,\mathrm{p1}\cdot \ll 2 \gg + \mathrm{q1}^2\right)^2}$$

$$\frac{C_A\,g^2\,\mathrm{q1}^{\mu 1}\,\left(\mathrm{p1}^{\nu 1} + \mathrm{q1}^{\nu 1}\right)\,\delta_{\mathrm{a1\,b1}}}{\mathrm{q1}^2\,\left(\mathrm{p1}^2 + 2\,\mathrm{p1}\cdot\mathrm{q1} + \mathrm{q1}^2\right)}$$

Mathematica package *Symb2Alg*: Transforms output of *DoDSE* into algebraic expressions.
Depending on Feynman rules compatible with *FeynCalc*.

Dyson-Schwinger equations (DSEs) for investigating QCD

Infinitely large tower of equations

Equations of motion of Green functions

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Equations of motion of Green functions

Pros:

- Exact equations
  $\longrightarrow$ non-perturbative regime accessible
- Continuum, different scales accessible
  $\longrightarrow$ complement lattice method

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Cons?:

- Truncations
- Gauge-dependent

Dyson-Schwinger equations (DSEs) for investigating QCD

Infinitely large tower of equations

Equations of motion of Green functions

Pros:

- Exact equations
  $\longrightarrow$ non-perturbative regime accessible
- Continuum, different scales accessible
  $\longrightarrow$ complement lattice method

Cons?:

- Truncations (not for all tasks)
- Gauge-dependent
  $\longrightarrow$ Exploit advantages of different gauges

The maximally Abelian gauge (MAG)

Dual superconductor picture of confinement

Magnetic monopoles condense $\rightarrow$ squeeze electric flux into flux tubes.

Ezawa, Iwazaki, PRD 25 (1981): Hypothesis of Abelian dominance (Abelian part should dominate the infrared part of the theory, since monopoles live in Abelian part of algebra.)

Gauge field: $A_\mu = A_\mu^r T^r, \quad r = 1, \ldots, N^2 - 1$
T^r is the generator of the gauge group $SU(N)$

Abelian subalgebra: $[T^i, T^j] = 0$, can be written as diagonal matrices
Split the gauge field: Abelian/Diagonal and non-Abelian/off-diagonal fields

$$A_\mu = \boldsymbol{A_\mu^i T^i} + \boldsymbol{B_\mu^a T^a}, \quad i = 1, \ldots, N-1, \quad a = N, \ldots, N^2 - 1$$

Gauge fixing to the MAG

$$A_\mu = \boldsymbol{A}^i_\mu \boldsymbol{T}^i + \boldsymbol{B}^a_\mu \boldsymbol{T}^a, \quad i = 1, \ldots, N-1, \quad a = N, \ldots, N^2-1$$

E.g. $T^1 = \frac{1}{2}\lambda^3$, $T^2 = \frac{1}{2}\lambda^8$ for $SU(3)$.

Fix the gauge such that norm of off-diagonal gluon field $\boldsymbol{B}$ is minimized:

$$D^{ab}_\mu \boldsymbol{B}^b_\mu = (\delta_{ab}\partial_\mu - g\, f^{abi} \boldsymbol{A}^i_\mu)\boldsymbol{B}^b_\mu = 0$$

Symmetry of diagonal part: $U(1)^{N-1}$
Fix gauge of diag. gluon field $\boldsymbol{A}$ by Landau gauge condition: $\partial_\mu \boldsymbol{A}_\mu = 0$.

NB: $SU(2)$ has only one diagonal field $\Rightarrow$ only one type of color structure constant: f^{abi}, no f^{abc}.

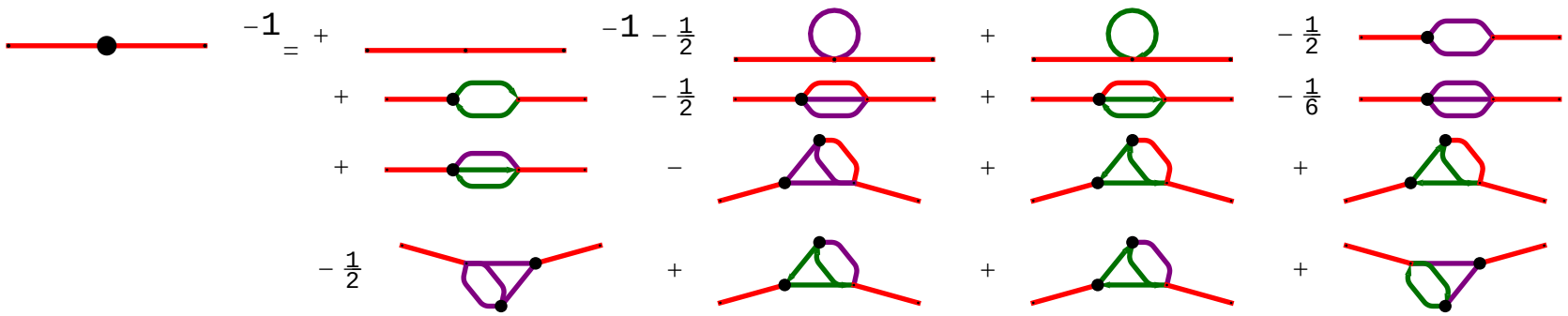
Peculiarities of the maximally Abelian gauge for $SU(2)$

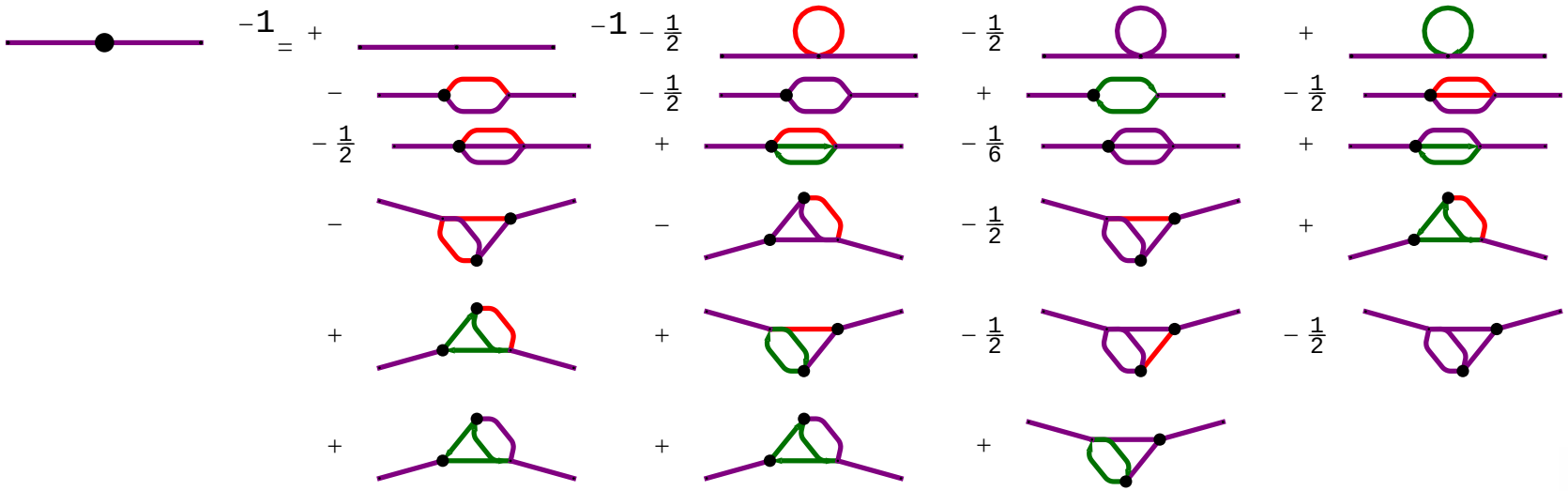
- Yang-Mills vertices split: $\boldsymbol{ABB}$, $\boldsymbol{AABB}$, $\boldsymbol{BBBB}$.
- Non-linear gauge fixing condition (depends on $\boldsymbol{A}$) $\rightarrow$ $\boldsymbol{Acc}$, $\boldsymbol{AAcc}$, $\boldsymbol{BBcc}$.
- Renormalizability requires an additional quartic ghost interaction $\rightarrow$ $\boldsymbol{cccc}$.
- Ghosts also split into diagonal and off-diagonal parts, but diagonal ghosts decouple (diagonal ghost equation).
- Two gauge fixing parameters: $\alpha_A = 0$ (Landau gauge), α_B.

Note: For $SU(N)$ there are four interactions more (due to f^{abc}) $\rightarrow$ more DSEs with more terms.

This plethora of interactions makes the equations much more intricate than in Landau gauge. To consider all possible solutions an improved method is necessary.

DSEs of the MAG





Landau gauge and maximally Abelian gauge: Current Status

	Landau gauge	MAG
propagators	A, c	A, B, c
interactions	AAA, Acc; $AAAA$	ABB, Acc; $AABB$, $AAcc$, $BBcc$, $BBBB$, $cccc$
Gribov region	bounded in all directions	bounded in off-diagonal and unbounded in diagonal direction [Capri et al, PRD 79 (2009)]
decoupling sol.	lattice, refined Gribov-Zwanziger framework and functional equations	lattice [Mendes et al., arXiv:0809.3741], ref. Gribov-Zwanziger framework [Capri et al., PRD 77 (2008)]; $SU(2)$ only
scaling solution	funct. eqs., lattice in 2d [von Smekal, Alkofer, Hauck, PRL 79 (1997); Pawlowski et al., PRL 93 (2004); Maas, PRD 75 (2007)]	this talk ($SU(N)$) [M.Q.H., Schwenzer, Alkofer, arXiv:0904.1873]

Loop integrals for low external momenta

We want to know how a vertex function behaves, when the external momenta approach 0 simultaneously:

$$\Gamma(p_1, p_2, \ldots) \text{ for } p_i \to 0$$

Generic propagator

$$L_{(\mu\nu)} \cdot \frac{D(p)}{p^2},$$

assume power law behavior at low p

$$D^{IR}(p) = A \cdot (p^2)^{\delta}$$

IR exponent

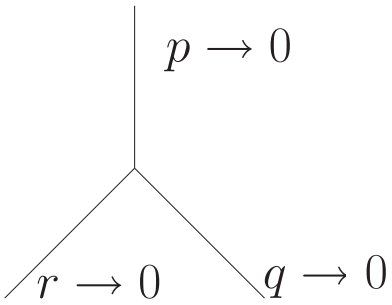
Example: Ghost propagator

$$\int \frac{d^d q}{(2\pi)^d} P_{\mu\nu} \frac{D^{AA}(q)}{q^2} \Gamma^{Acc,0}(p,q) \frac{D^{cc}(p-q)}{(p-q)^2} \Gamma^{Acc}(p,q)$$

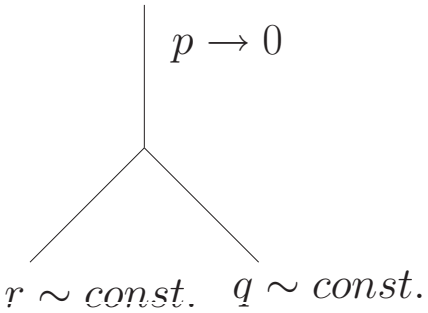
Integrals are dominated by $1/(p-q)^2 \to$ use IR expressions for all quantities.

Vertices also assume power law behavior [Alkofer, Fischer, Llanes-Estrada, PLB 611 (2005)] (skeleton expansion).

More infrared exponents



Up to now it was assumed that all momenta of a Green function go to zero simultaneously $\to$ uniform IR exponents.



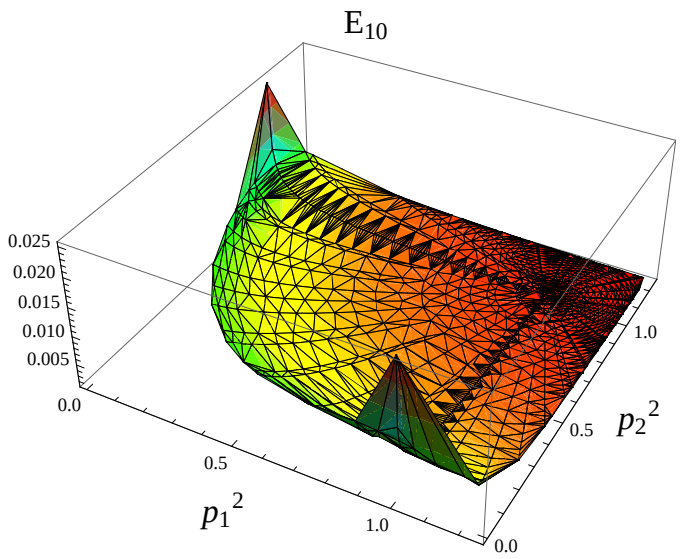
For a three-point function there is one additional possibility: Only one momentum goes to zero $\to$ kinematic IR exponents [Alkofer, M.Q.H., Schwenzer, 0801.2762].

There is a non-uniform dependence on the momenta [Alkofer, M.Q.H., Schwenzer, EPJC to be pub.]; ex. on next slide.

More infrared exponents: Examples

The first order contribution of the three-gluon vertex in the IR (ghost triangle) can be described by 10 dressing functions:

$$\Gamma^{\Delta}_{\mu\nu\rho}(p_1, p_2, p_3) = \sum_{i=1}^{10} E_i(p_1^2, p_2^2, p_3^2; \kappa; d)\tau^i_{\mu\nu\rho}(p_1, p_2, p_3).$$



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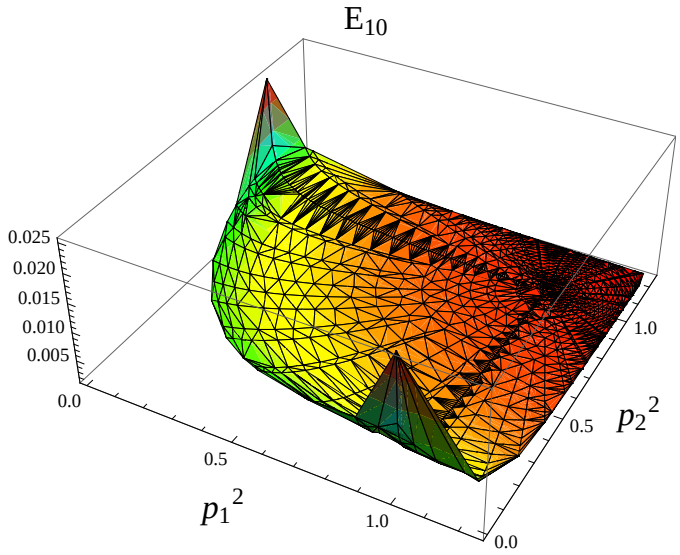
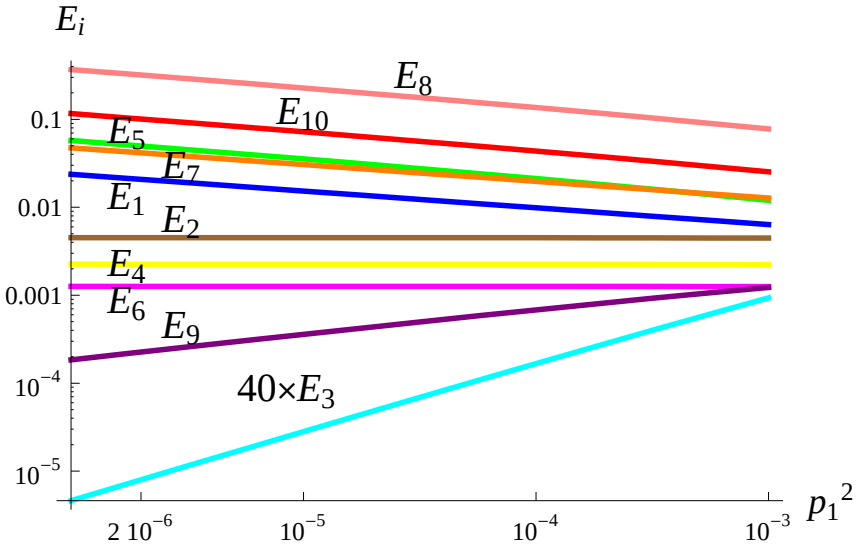


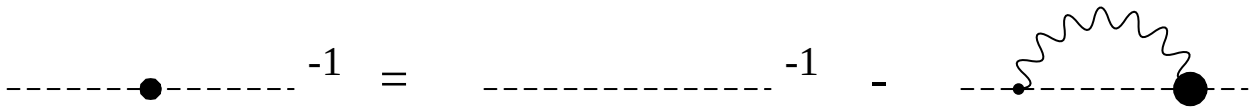


These singularities only appear in the longitudinal parts [Alkofer, M.Q.H., Schwenzer, EPJC to be pub.] $\Rightarrow$ play no role in Landau gauge [Fischer, Pawlowski, 0810.1987, 0903.2193].

Power counting

In the remainder only uniform IR exponents.

- The ghost propagator DSE:



- Plug in power law ansätze for dressing functions in the IR (In Landau gauge the ghost-gluon vertex has an IR constant dressing.):

$$\left(\frac{B\cdot(p^2)^\beta}{p^2}\right)^{-1} \sim \int \frac{d^dq}{(2\pi)^d}\, P_{\mu\nu}\, \frac{A\cdot(q^2)^\alpha}{q^2}\, \frac{B\cdot((p-q)^2)^\beta}{(p-q)^2}\, (p-q)_\mu q_\nu$$

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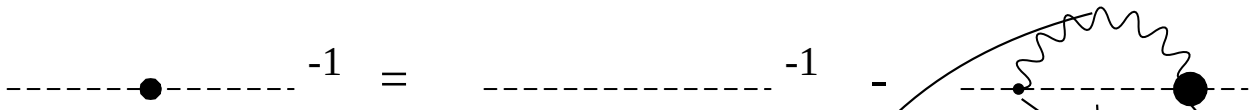

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- Only one momentum scale
  $\longrightarrow$ simple power counting is possible $\longrightarrow$ scaling relation:
- $1-\beta = \frac{d}{2} + \alpha - 1 + \beta - 1 + \frac{1}{2} + \frac{1}{2} \Longrightarrow -2\beta = \alpha + \frac{d}{2} - 2$

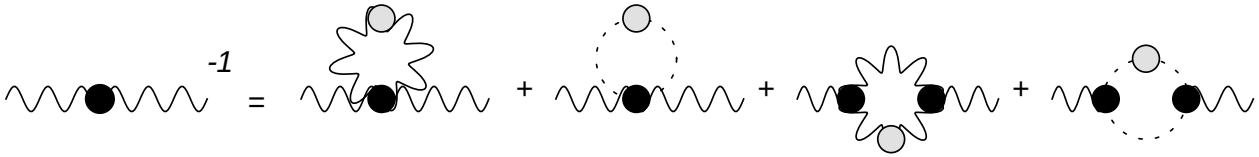
Functional Renormalization Group

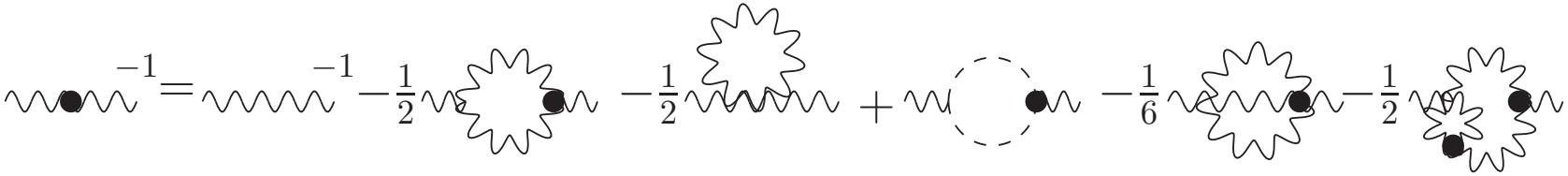
Functional equations similar to DSEs, but with decisive differences:

- only 1-loop diagrams
- ALL quantities dressed
- (appearance of regulator)

Renormalization group equations (RGEs) are "differential DSEs".

Compare RGE and DSE of gluon propagator:

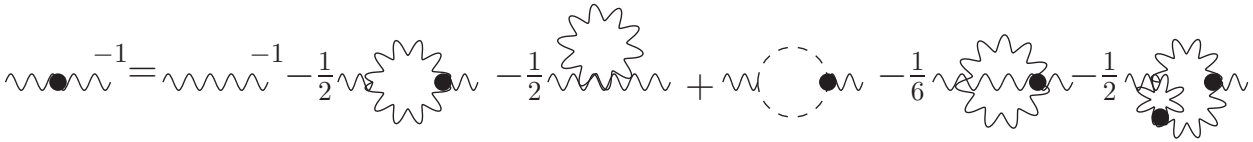




Systems of inequalities

- For every diagram the IR can be written down.
- At least the IRE of one diagram must equal the IRE of the vertex function on the lhs.
- No diagram can be more IR divergent than the vertex function on the lhs $\rightarrow \delta_{lhs} \leq \delta_{rhs}$.
- Not knowing which diagram is leading on the rhs, we can write inequalities from all diagrams.

$$-\delta_{gl} = \min(\underbrace{0}_{\text{bare prop.}}, \underbrace{2\delta_{gl} + \delta_{3g}}_{\text{gh loop}}, \underbrace{2\delta_{gh} + \delta_{gg}}_{\text{gl loop}}, \underbrace{\delta_{gl}}_{\text{tadpole}}, \underbrace{4\delta_{gl} + 2\delta_{3g}}_{\text{squint}}, \underbrace{3\delta_{gl} + \delta_{4g}}_{\text{sunset}})$$



Infrared exponent for an arbitrary diagram

Having so many diagrams, isn't there a shorter way than writing all expressions down explicitly?

Arbitrary Diagram v

Numbers of vertices and propagators related $\Rightarrow$ possible to get a formula for the IR exponent by pure combinatorics.

Function of:

- propagator IR exponents δ_{Φ_i}
- number of external legs m^{Φ_i}
- number of vertices.

$$\delta_v = \boxed{-\frac{1}{2}\sum_i m^{\Phi_i}\delta_{\Phi_i}} + \\ + \sum_i (\#\text{ of dressed vertices})_i C_1^i + \sum_i (\#\text{ of bare vertices})_i C_2^i$$

Only depends on the external legs $\longrightarrow$ equal for all diagrams in a DSE/RGE [M.Q.H., Schwenzer, Alkofer, arXiv:0904.1873].

[Similar formula with slightly different arguments: Fischer, Pawlowski, arXiv:0903.2193]

Relevant inequalities

ALL relevant inequalities can be written down in closed form:

	valid for	derived from
$C_1^i = \delta_{vertex} + \frac{1}{2} \sum_{\substack{\text{legs } j \text{ of} \\ \text{vertex}}} \delta_j \geq 0$	dressed vertices	RGEs
$C_2^i = \frac{1}{2} \sum_{\substack{\text{legs } j \text{ of} \\ \text{prim. div.} \\ \text{vertex}}} \delta_j \geq 0$	prim. divergent vertices	DSEs/RGEs

Some inqualities are contained within others.
E. g. in MAG: $\delta_B \geq 0$ and $\delta_c \geq 0$ render $\delta_B + \delta_c \geq 0$ useless.

Only some inequalities are restrictive.

NB: These inequalities explicitly show that the skeleton expansion used in previous studies is a consistent expansion. However, the skeleton expansion is now obsolete.

Maximally infrared divergent solution

The inequalities derived from DSEs and RGEs allow to derive a lower bound on the IREs.

$$C_1^i \geq 0, \qquad C_2^i \geq 0.$$

IR solution:

$$\delta_v = -\frac{1}{2}\sum_i m^{\Phi_i}\delta_{\Phi_i} + \sum_i (\#\text{ dr. vert.})_i C_1^i + \sum_i (\#\text{ bare vert.})_i C_2^i.$$

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$$C_1^i \geq 0, \qquad C_2^i \geq 0.$$

$\Rightarrow$ Maximally IR divergent solution:

$$\boxed{\delta_{v,max} = -\frac{1}{2}\sum_i m^{\Phi_i}\delta_{\Phi_i}} + \cancel{\sum_i (\#\text{ dr. vert.})_i C_1^i} + \cancel{\sum_i (\#\text{ bare vert.})_i C_2^i}\,.$$

IR scaling solutions

A general analysis of propagator DSEs yields that at least one inequality from a prim. divergent vertex has to be saturated. $\Leftarrow$ one bare vertex in DSEs, none in RGEs

Consistency condition DSEs $\leftrightarrow$ RGEs [Fischer, Pawlowski, PRD 75 (2007)].


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One inequality saturated $\Rightarrow$ one primitively divergent vertex does not acquire an IR enhanced dressing.

The non-enhancement of at least one primitively divergent vertex is now established for all scaling type solutions. [MQH, Schwenzer, Alkofer, arXiv:0804.1873]

IR scaling solution of Landau gauge

Up to some time ago

- Ghost-gluon vertex used as bare vertex (INPUT); connected to the non-renormalization of the vertex.
- Skeleton expansion to determine behavior of vertices.

- Here it is shown that this is a necessary condition for the scaling solution.
- Skeleton expansion obsolete.

IR scaling solutions for Landau gauge and MAG

Inequalities:

Landau gauge	MAG
$\delta_{gl} \geq 0$	$\delta_B \geq 0,\ \delta_c \geq 0$
$\frac{1}{2}\delta_{gl} + \delta_{gh} \geq 0$	$\delta_A + \delta_B \geq 0,\ \delta_A + \delta_c \geq 0$

Test the consequences, if some of the inequalities are saturated:

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Test the consequences, if some of the inequalities are saturated:

- Saturation in first row corresponds to trivial solution: $\delta_i = 0$ ($\rightarrow$ perturbation theory)
- Second row yields scaling relations: $\delta_{gl} = -2\delta_{gh} = 2\kappa_{LG}$ and $\delta_B = \delta_c = -\delta_A = \kappa_{MAG}$
  - New scaling solution for MAG [M.Q.H., Schwenzer, Alkofer, arXiv:0904.1873].
  - Known IR scaling solution of Landau gauge [von Smekal, Hauck, Alkofer, PRL (1997)]. But: no skeleton expansion, no external input for ghost-gluon vertex.

IR Scaling solutions for other gauges

The analysis can be used also for other gauges. Beware: This corresponds to a naive application!

Linear covariant gauges	Ghost-antighost symmetric gauges
scaling solution only, if the longitudinal part of the gluon propagator gets dressed (STI?)	quartic ghost interaction $\rightarrow \delta_{gh} \geq 0$ $\rightarrow$ with non-negative IREs only the trivial solution can be realized

$\Longrightarrow$

- Either the existence of a scaling solution is something special (?) or
- a more refined analysis is needed in these cases.

Details about the MAG solution

$$-\delta_A = \delta_B = \delta_c := \kappa \geq 0$$

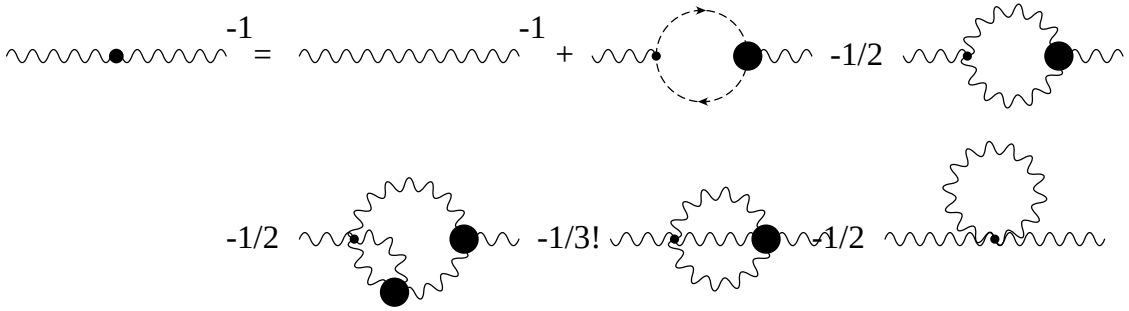
- Diagonal gluon propagator is IR enhanced ($\delta_A \leq 0$). $\Rightarrow$ Supports hypothesis of Abelian dominance.
- Off-diagonal propagators are IR suppressed.
- Two-loop terms are leading.

The scaling solution for the MAG differs in several qualitative and technical aspects from the Landau gauge solution:

- Different qualitative behavior of ghosts.
- Different structure of IR leading terms $\rightarrow$ new method for numerical solutions required.
- Different DSEs for $SU(2)$ and $SU(3)$ $\rightarrow$ different solutions?

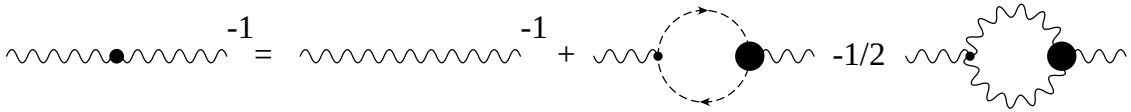
Numerical solution

In Landau gauge trunction "straight forward": keep one-loop terms (consistent UV behavior, contain IR leading term)



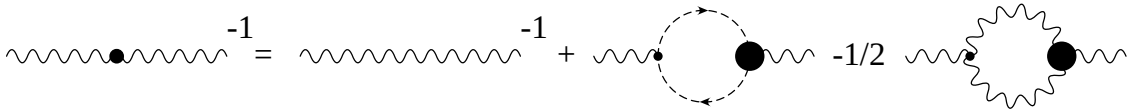
Numerical solution

In Landau gauge trunction "straight forward": keep one-loop terms (consistent UV behavior, contain IR leading term)

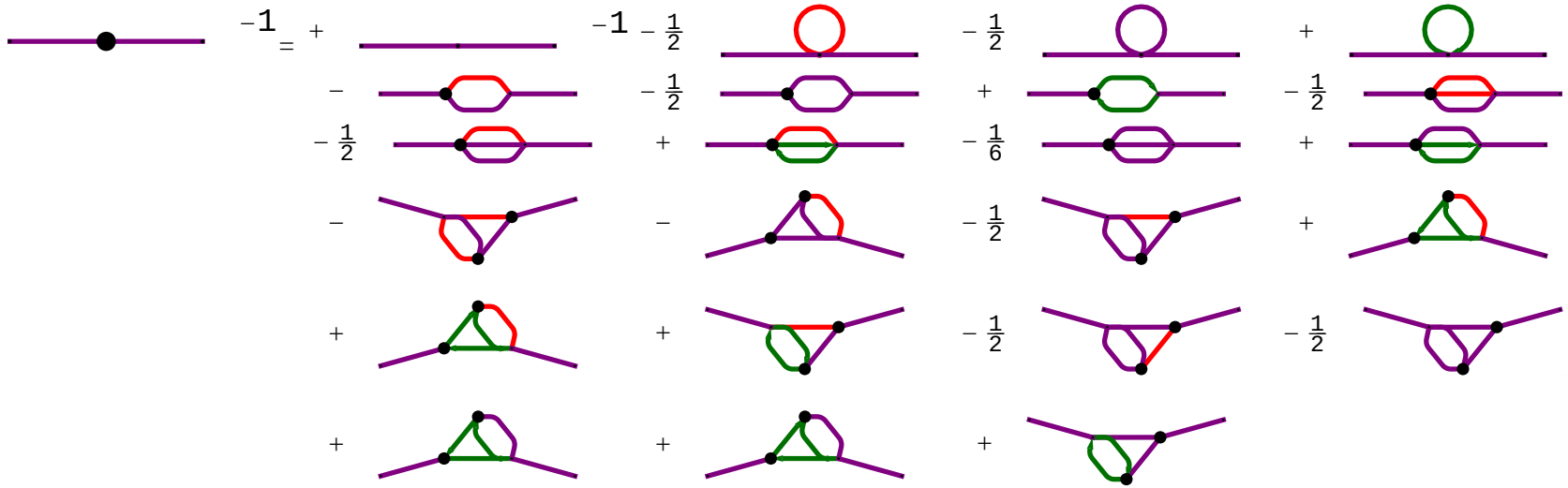


Numerical solution

In Landau gauge trunction "straight forward": keep one-loop terms (consistent UV behavior, contain IR leading term)



In MAG: two-loop terms leading → for consistent UV behavior keep ALL two-loop terms = no truncation



The MAG in $SU(3)$

In general $SU(N)$ there are more interactions than included above.
$\rightarrow$ Different solution for "physical system", i. e. $SU(3)$?

4 additional vertices: ***BBB***, ***Bcc***, ***ABBB***, ***ABcc***
Constraints:

$$\frac{3}{2}\delta_B \geq 0, \qquad \frac{1}{2}\delta_B + \delta_c \geq 0,$$
$$\frac{1}{2}\delta_A + \frac{3}{2}\delta_B \geq 0, \qquad \frac{1}{2}\delta_A + \frac{1}{2}\delta_B + \delta_c \geq 0$$

Solution for $SU(N > 2)$ = solution for $SU(2)$

- Constraints already contained in "old" system $\rightarrow$ nothing new, solution still valid.
- No new solutions possible $\rightarrow$ unique solution.

Summary

- Improved method establishes consistency condition between DSEs and RGEs: One vertex gets not IR enhanced.
- Skeleton expansion obsolete.
- Qualitative solution for whole tower of functional equations.
- High number of interactions can be handled, because it is not necessary to write down all equations explicitly.
- Derivation of method technical, but it allows a straightforward application based only on the type of interactions in the Lagrangian.
- Method allows a first assessment what a scaling solution might look like. $\longrightarrow$ Input for a complete numeric calculation.

Conclusions on MAG

- The MAG may possess an IR scaling solution.
- This solution is in support of the hypothesis of Abelian dominance, because the diagonal gluon propagator is IR enhanced and thereby the dynamics in the IR are dominated by the diagonal gluon.
- Relation to monopole condensation has to be clarified.
- Although the DSEs are more complicated for general $SU(N > 2)$, the qualitative behavior is the same as in $SU(2)$.

The existence of the IR scaling solution in the MAG has to be verified by a numerical solution of the DSEs, which is more involved than in Landau gauge. $\rightarrow$ Task for the future.