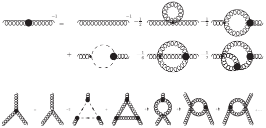
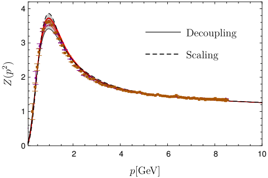
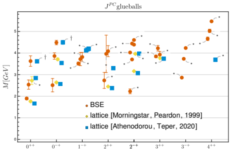
# Glueballs from functional equations








Markus Q. Huber

Institute of Theoretical Physics

Giessen University

MQH, Phys.Rev.D 101, arXiv:2003.13703

MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C 80, arXiv:2004.00415

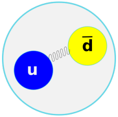
A Virtual Tribute to *Quark Confinement and the Hadron Spectrum* 2021, virtually in Stavanger, Norway, Aug. 6, 2021

FWF Der Wissenschaftsfonds.

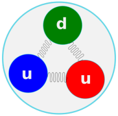
JUSTUS-LIEBIG-UNIVERSITÄT GIESSEN

DFG Deutsche Forschungsgemeinschaft

# Bound states in QCD

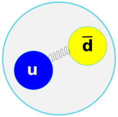



Mesons




Baryons

# Bound states in QCD

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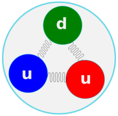


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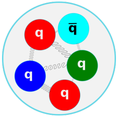
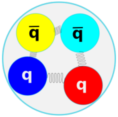
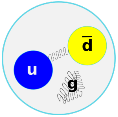



Pentaquarks — First observations 2015 (LHCb)




Tetraquarks — Increasing number of confirmed states. Bound state equations perspective: [Eichmann, Fischer, Heupel, Santowsky, Wallbott '20]




Hybrids


Glueballs — States of pure 'radiation'

# Glueball observations

Experimental candidates, but situation not conclusive.

Scalar glueball: $0^{++}$, mixing with scalar isoscalar mesons

Candidate reaction: $J/\psi \to \gamma + 2g$

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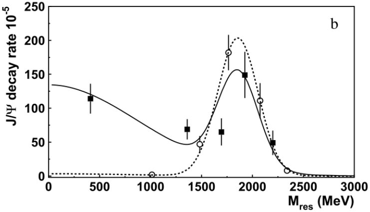
Recent analysis of BESIII data [Sarantsev, Denisenko, Thoma, Klempt '21]:

$$M = 1865 \pm 25^{+10}_{-30}\,\text{MeV},$$
$$\Gamma = 370 \pm 50^{+30}_{-20}\,\text{MeV}$$




→ Talk Klempt

# Glueball calculations

## Yang-Mills theory

- "Isolated" problem: only gluons
- Clean picture: well-established lattice results

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QCD glueballs: mixing with quarks

Unquenching on the lattice [Gregory et al. ’12]:

- Much higher statistics required (poor signal-to-noise ratio)
- Continuum extrapolation and inclusion of fermionic operators still to be done
- Mixing with $\bar{q}q$ challenging
- Tiny (e.g., $0^{++}$, $2^{++}$) to moderate unquenching effects (e.g., $0^{-+}$) found
- $m_\pi = 360\,\text{MeV}$

# Glueball calculations

## Yang-Mills theory

- "Isolated" problem: only gluons
- Clean picture: well-established lattice results
- Functional methods: High quality input available for bound state equations

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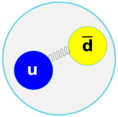
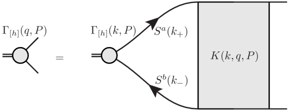
# Hadrons from bound state equations

Example: Meson



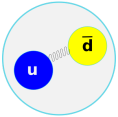



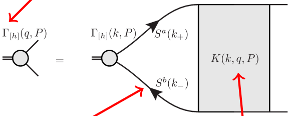

Integral equation: $\Gamma(q,P) = \int dk\, \Gamma(k,P)\, S(k_+)\, S(k_-)\, K(k,q,P)$

# Hadrons from bound state equations

Bethe-Salpeter amplitude

Example: Meson

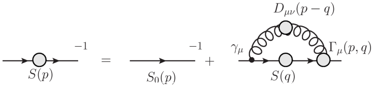



Integral equation: $\Gamma(q,P) = \int dk\, \Gamma(k,P)\, S(k_+)\, S(k_-)\, K(k,q,P)$

Ingredients:

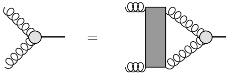
- Quark propagator $S$




Nonperturbative diagram: full momentum dependent dressings $\rightarrow$ numerical solution

- Interaction kernel $K$
- Constrained by symmetries

# Glueball BSE




Need  and , solve for . $\rightarrow$ Mass

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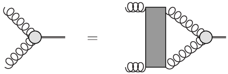

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Not quite. . .

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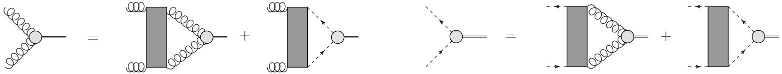

Gluons couple to ghosts $\rightarrow$ Include 'ghostball'-part.

(First step: no quarks
$\rightarrow$ Yang-Mills theory)

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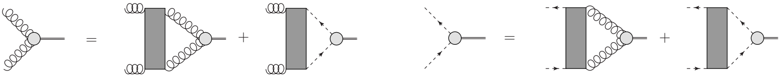

Gluons couple to ghosts → Include 'ghostball'-part.

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Need ,  and 4× , solve for  and . → Mass

**Construction of kernel**

Consistency with input: Apply same construction principle.

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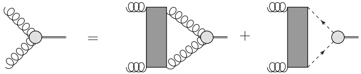

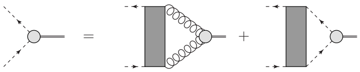

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Previous BSE calculations for glueballs:

- [Meyers, Swanson '13]
- [Sanchis-Alepuz, Fischer, Kellermann, von Smekal '15]
- [Souza et al. '20]
- [Kaptari, Kämpfer '20]

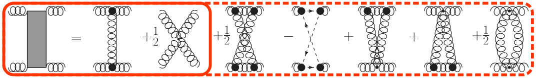
⇒ Input is important for quantitative predictive power!

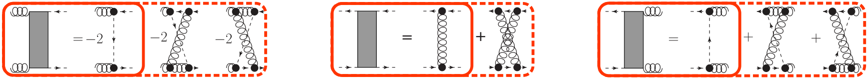
[MQH, Fischer, Sanchis-Alepuz '20]

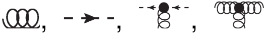
# Kernel construction

From 3PI effective action truncated to three-loops:

[Fukuda '87; McKay, Munczek '89; Sanchis-Alepuz, Williams '15; MQH, Fischer, Sanchis-Alepuz '20]



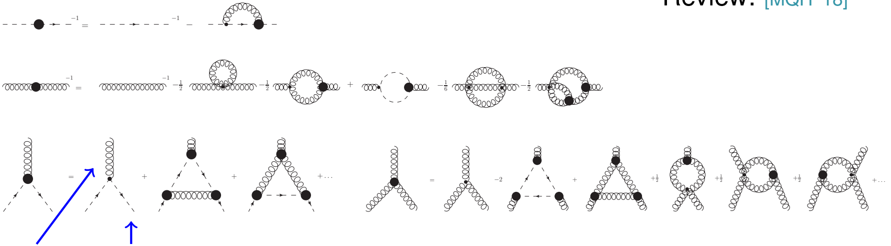




→ Need .

- Some diagrams vanish for certain quantum numbers.
- Full QCD: Same for quarks → Mixing with mesons.

# Equations of motion from 3-loop 3PI effective action

Review: [MQH '18]



Gluon and ghost fields: Elementary fields of Yang-Mills theory in the Landau gauge

Self-contained system of equations with the scale as the only input.

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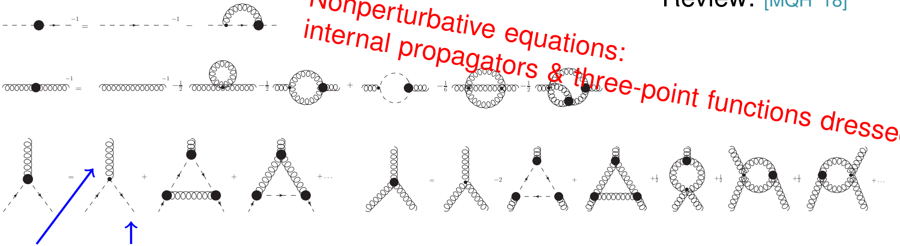
internal propagators & three-point functions dressed



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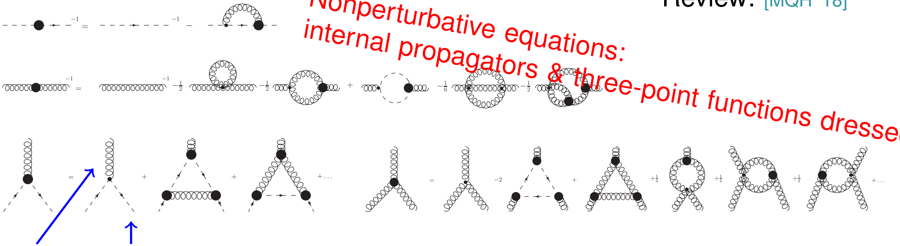

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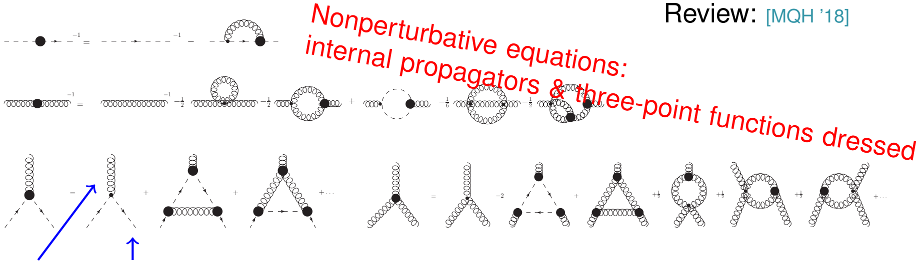



Gluon and ghost fields: Elementary fields of Yang-Mills theory in the Landau gauge

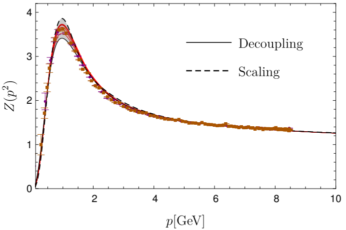
Self-contained system of equations with the scale as the only input.

Truncation→ 3-loop expansion of 3PI effective action [Berges '04]

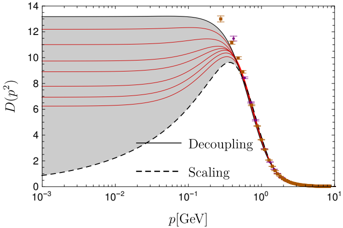
- 4 coupled integral equations with full kinematic dependence.
- Sufficient numerical accuracy required for renormalization.
- One- and two-loop diagrams [Meyers, Swanson '14; MQH '17; Eichmann, Pawlowski, Silva '21].

# Landau gauge propagators

Gluon dressing function:

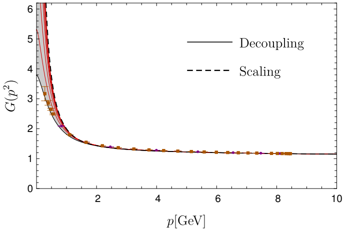



Gluon propagator:




- Family of solutions: Nonperturbative completions of Landau gauge [Maas '10]?
- Realized by condition on $G(0)$ [Fischer, Maas, Pawlowski '08; Alkofer, MQH, Schwenzer '08]
- Results here independent of $G(0)$

Ghost dressing function:

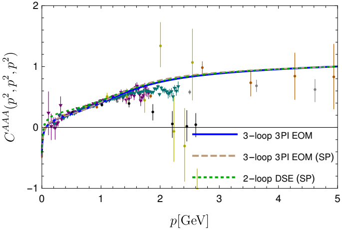



[Sternbeck '06; MQH '20]

# Concurrence of functional methods

Exemplified with three-gluon vertex.

3PI vs. 2-loop DSE:

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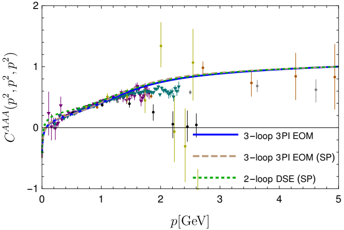


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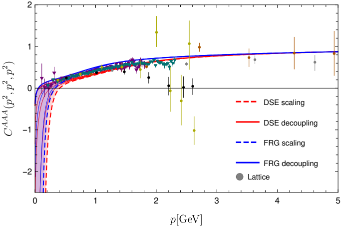
DSE vs. FRG:




[Cucchieri, Maas, Mendes ’08; Sternbeck et al. ’17; Cyrol et al. ’16; MQH ’20]

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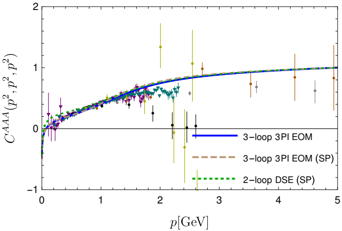


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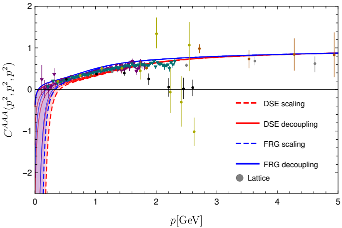
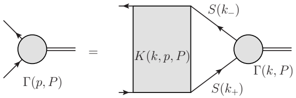



[Cucchieri, Maas, Mendes '08; Sternbeck et al. '17; Cyrol et al. '16; MQH '20]

## Beyond this truncation

- Further dressings of three-gluon vertex [Eichmann, Williams, Alkofer, Vujinovic '14]
- Effects of four-point functions [MQH '16, MQH '17, Corell et al. '18, MQH '18]

# Solving a BSE



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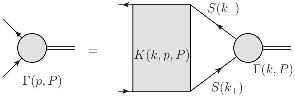


Consider the eigenvalue problem ($\Gamma$ is the BSE amplitude)

$$\mathcal{K}\cdot\Gamma(P) = \boldsymbol{\lambda(P)}\,\Gamma(P).$$

$\lambda(P^2) = 1$ is a solution to the BSE $\Rightarrow$ Glueball mass $P^2 = -M^2$

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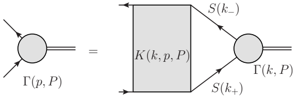


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Calculation requires quantities for

$$k_\pm^2 = P^2 + k^2 \pm 2\sqrt{P^2\,k^2}\cos\theta = -M^2 + k^2 \pm 2\,\boldsymbol{i}\,M\sqrt{k^2}\,\cos\theta.$$

$\Rightarrow$ Complex momentum arguments.

Direct calculation from functional methods possible, e.g., [Fischer, MQH '20].

$\rightarrow$ talk by Windisch

**Alternative**

Extrapolate $\lambda$ from $P^2 > 0$.

# Extrapolation of $\lambda(P^2)$

## Extrapolation method

- Extrapolation to time-like $P^2$ using Schlessinger's continued fraction method (proven superior to default Padé approximants) [Schlessinger '68]
- Average over extrapolations using subsets of points for error estimate

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$$f(x) = \frac{f(x_1)}{1 + \frac{a_1(x-x_1)}{1+\frac{a_2(x-x_2)}{1+\frac{a_3(x-x_3)}{\dots}}}}$$

Coefficients $a_i$ can determined such that $f(x)$ exact at $x_i$.

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Test extrapolation for solvable system:
Heavy meson [MQH, Sanchis-Alepuz, Fischer '20]

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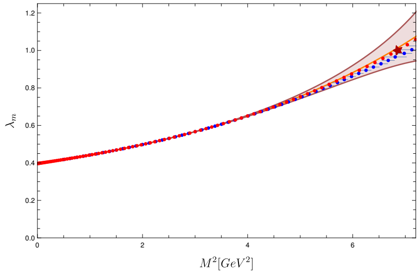

# Extrapolation of $\lambda(P^2)$ for glueballs

Higher eigenvalues: Excited states.

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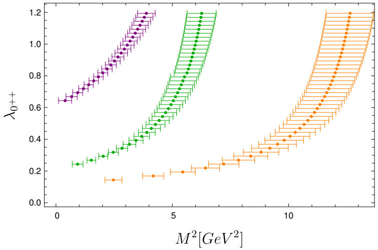
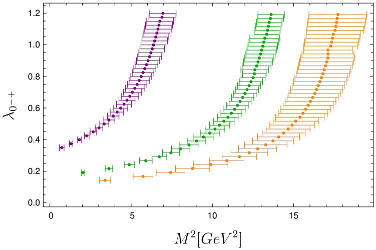
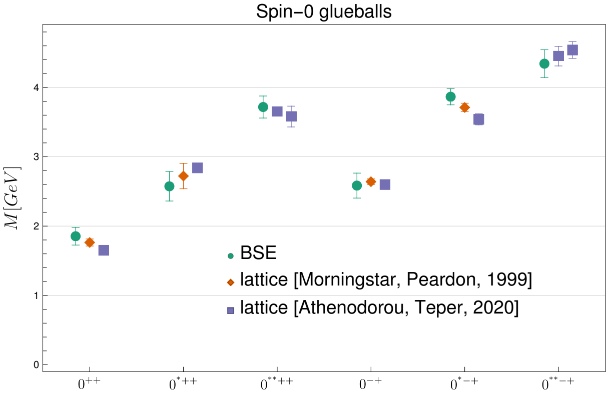



Physical solutions for $\lambda(P^2) = 1$.

# Glueballs masses for $0^{\pm +}$




Lattice $0^{**++}$: Conjectured based on irred. rep. of octahedral group

All results for $r_0 = 1/418(5)$ MeV.

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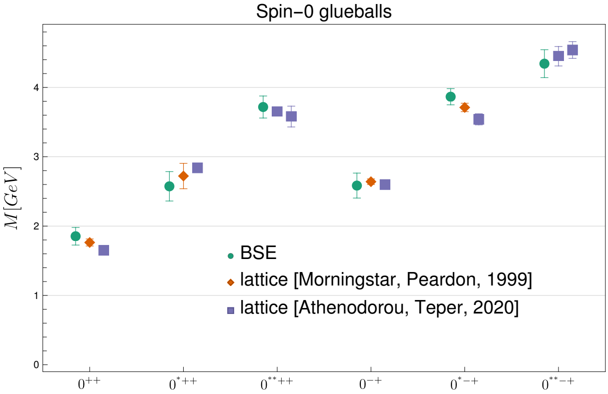


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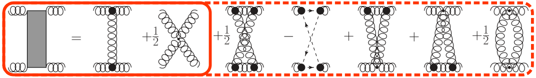
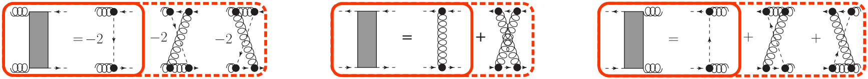
# Two-loop diagrams

Results from [MQH, Fischer, Sanchis-Alepuz ’20] were from one-loop terms only:





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# Two-loop diagrams

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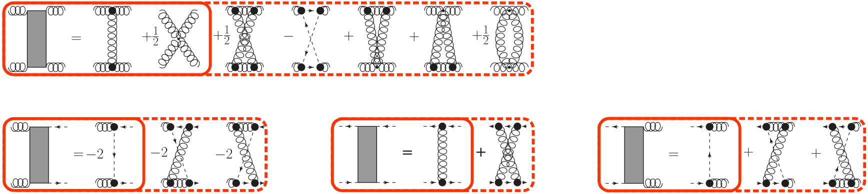

Fully self-consistent DSE/BSE truncation

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Drastic increase in computational resources, hence lower precision used.

Preliminary result for $0^{++}$, $0^{-+}$: No effect on mass.

# Glueball masses for $J^{\pm+}$

For higher spin, larger tensor bases: more tensors, more indices

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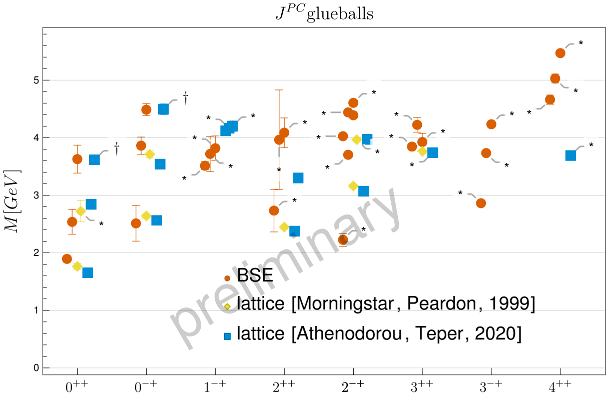


Lattice:
*: identification with some uncertainty
†: conjecture based on irred. rep of octahedral group

[MQH, Fischer, Sanchis-Alepuz, in preparation]

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Parameter-free determination of glueball masses from functional methods.

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Thank you for your attention.

# Glueballs as bound states

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$$D(x-y) = \langle O(x)O(y) \rangle$$

- $\rightarrow$ Lattice: Mass from this correlator by exponential Euclidean time decay.
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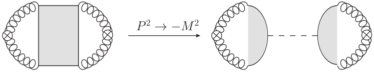
Put total momentum on-shell and consider individual 2-, 3- and 4-gluon contributions. $\rightarrow$ Each can have a pole at the glueball mass.

$A^4$-part of $D(x-y)$, total momentum on-shell:



# Charge parity

Transformation of gluon field under charge conjugation:

$$A^a_\mu \to -\eta(a) A^a_\mu$$

where

$$\eta(a) = \begin{cases} +1 & a = 1, 3, 4, 6, 8 \\ -1 & a = 2, 5, 7 \end{cases}$$

Color neutral operator with two gluon fields:

$$A^a_\mu A^a_\nu \to \eta(a)^2 A^a_\mu A^a_\nu = A^a_\mu A^a_\nu.$$

$\Rightarrow C = +1$

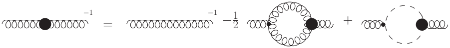
Negative charge parity, e.g.:

$$d^{abc} A^a_\mu A^b_\nu A^c_\rho \to - d^{abc} \eta(a)\eta(b)\eta(c) A^a_\mu A^b_\nu A^c_\rho = \\ - d^{abc} A^a_\mu A^b_\nu A^c_\rho.$$

Only nonvanishing elements of the symmetric structure constant $d^{abc}$: zero or two indices equal to 2, 5 or 7.

# Landau gauge propagators in the complex plane

Simpler truncation:



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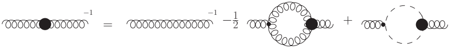


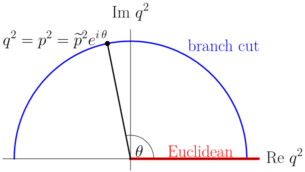


$\rightarrow$ Opening at $q^2 = p^2$.

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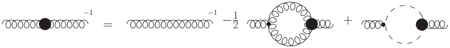
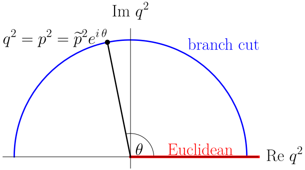
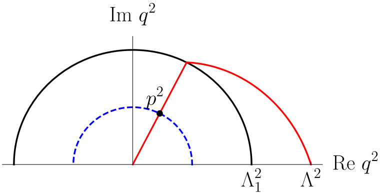





$\rightarrow$ Opening at $q^2 = p^2$.

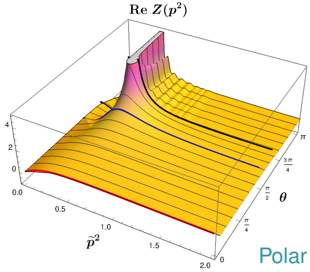
Appearance of branch cuts for complex momenta forbids integration directly to cutoff.

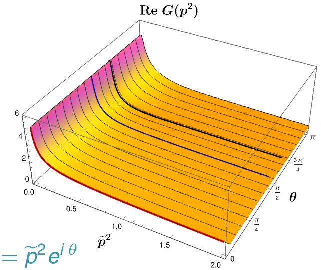
Deformation of integration contour necessary [Maris '95]. Recent resurgence: [Alkofer et al. '04; Windisch, MQH, Alkofer, '13; Williams '19; Miramontes, Sanchis-Alepuz '19; Eichmann et al. '19, . . . ]

# Landau gauge propagators in the complex plane

Ray technique for self-consistent solution of a DSE:

[Strauss, Fischer, Kellermann; Fischer, MQH '20].



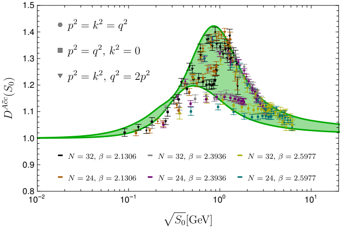

Polar coordinates: $p^2 = \widetilde{p}^2 e^{i\,\theta}$

- Current truncation leads to a pole-like structure in the gluon propagator.
- Analyticity up to 'pole' confirmed by various tests (Cauchy-Riemann, Schlessinger, reconstruction)
- No proof of existence of complex conjugate poles due to simple truncation.

[Fischer, MQH '20]

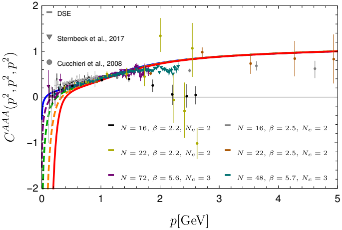
# Landau gauge vertices

Ghost-gluon vertex:




[Maas '19; MQH '20]

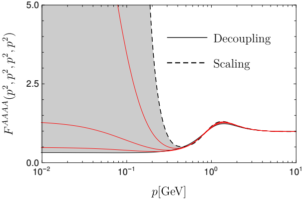
Three-gluon vertex:




[Cucchieri, Maas, Mendes '08; Sternbeck et al. '17; MQH '20]

- Nontrivial kinematic dependence of ghost-gluon vertex
- Simple kinematic dependence of three-gluon vertex
- Four-gluon vertex from solution

Four-gluon vertex:

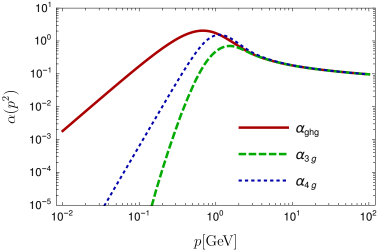



[MQH '20]

# Some properties of the Landau gauge solution

[MQH '20]

- Slavnov-Taylor identities (gauge invariance): Vertex couplings agree down to GeV regime



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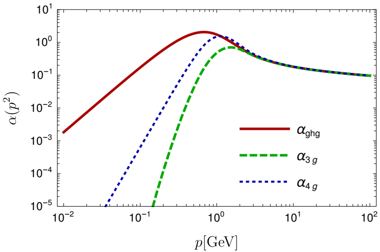


- Renormalization: First parameter-free subtraction of quadratic divergences
  $\Rightarrow$ One unique free parameter (family of solutions)