Dyson-Schwinger Equations in QCD

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SIC!QFT

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Why use DSEs?

Methods for investigating QCD:

- UV: perturbation series
- IR: effective theories and models
- Lattice: IR- and UV-cutoffs

And functional approaches (DSEs, ERGEs, nPIs, . . .)?

- Valid in all momentum regions
  - Arbitrary low momenta
  - Arbitrary high momenta
  - Mid-momentum region
- Only one formalism

Overview of Functional Approaches

Advantages

- Non-perturbative methods
  - Effects that are not included in perturbation theory
  - IR: study of confinement and chiral symmetry breaking
- UV: coincide with perturbation series
- Continuum methods (no IR- and UV-cutoffs as on the lattice) $\longrightarrow$ Supplement lattice techniques

Disadvantages

- Infinitely many equations $\longrightarrow$ Truncations (restrictions) needed
- Gauge dependent $\longrightarrow$ different equations for every gauge

Overview of Dyson-Schwinger Equations

History

- Basic works by F.J. Dyson (1949) and J.S. Schwinger (1951)
- In the course of time DSEs gave different pictures of QCD depending on level of truncation (more later)
- During the last ten years growing community

- Equations of motion of a quantum field theory
- System of (infinitely many) coupled integral equations
  $\longrightarrow$ "tower of DSEs"

Constituents of DSEs

Basic quantities: n-point functions

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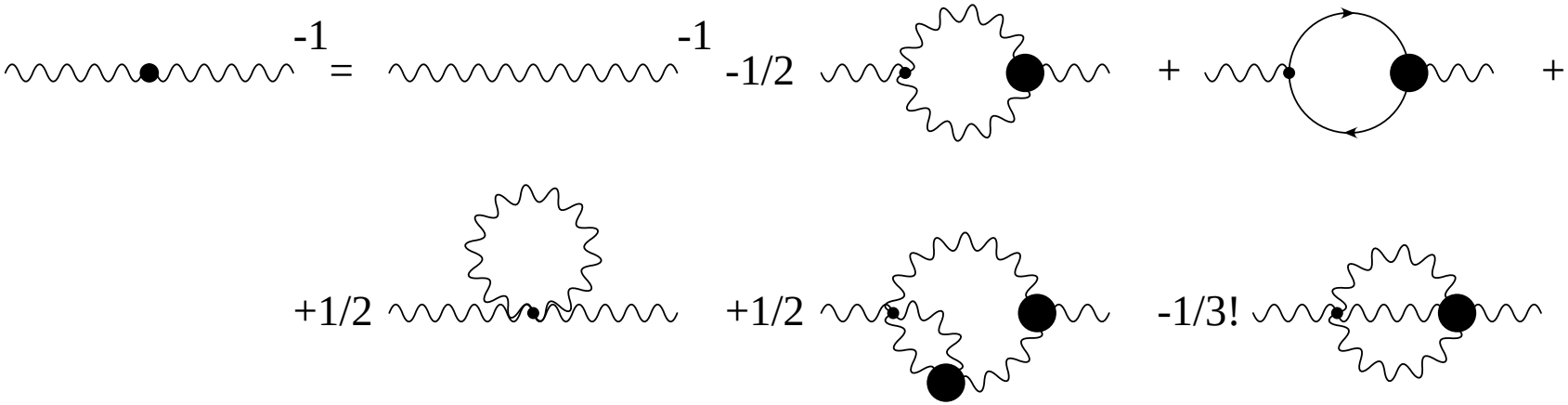
Gluon propagator DSE:

$$
\begin{aligned}
D^{-1}{}^{ab}_{\mu\nu}(p) = {} & Z_3\, D^{-1}_{(0)}{}^{ab}_{\mu\nu}(p) \\
& - Z_1\, \frac{1}{2} \int d^4q\, d^4k\, \Gamma^{(0)}{}^{acd}_{\mu\alpha\beta}(p,-q,-k)\, D^{de}_{\beta\gamma}(k)\, D^{cf}_{\alpha\delta}(q)\, \Gamma^{bef}_{\nu\gamma\delta}(-p,k,q) \\
& + Z_4\, \frac{1}{2} \int d^4q_1\, d^4q_2\, \Gamma^{(0)}{}^{abcd}_{\mu\nu\alpha\beta}(p,-p,-q_1,q_2)\, D^{cd}_{\alpha\beta}(q_1) \\
& - Z_4\, \frac{1}{6} \int d^4k_1\, d^4k_2\, d^4k_3\, \Gamma^{(0)}{}^{acde}_{\mu\alpha\beta\gamma}(p,-k_1,-k_2,-k_3) \\
& \quad \times D^{cm}_{\alpha\lambda}(k_1)\, D^{dl}_{\beta\sigma}(k_2)\, D^{ek}_{\gamma\rho}(k_3)\, \Gamma^{bklm}_{\nu\rho\sigma\lambda}(p,k_3,k_2,k_1) \\
& + Z_4\, \frac{1}{2} \int d^4k_1\, d^4k_2\, d^4k_3\, d^4k_4 \\
& \quad \times \Gamma^{(0)}{}^{acde}_{\mu\alpha\beta\gamma}(p,-k_1,-k_2,-k_3)\, D^{ck}_{\alpha\rho}(k_1)\, D^{dm}_{\beta\lambda}(k_2)\, D^{ep}_{\gamma\delta}(k_3) \\
& \quad \times \Gamma^{klm}_{\rho\sigma\lambda}(k_1,-k_4,k_2)\, D^{lq}_{\sigma\kappa}(k_4)\, \Gamma^{bpq}_{\nu\delta\kappa}(-p,k_3,k_4) \\
& - Z_{1F} \int d^4q\, d^4k\, \Gamma^{(0)}{}^{a}_{\mu}(p,q,k)\, S(-q)\, S(k)\, \Gamma^{b}_{\nu}(-p,k,q)
\end{aligned}
$$

Constituents of DSEs

Basic quantities: n-point functions

Gluon propagator DSE:



→ Often diagrams are used for representation.

Constituents of DSEs

n-point functions

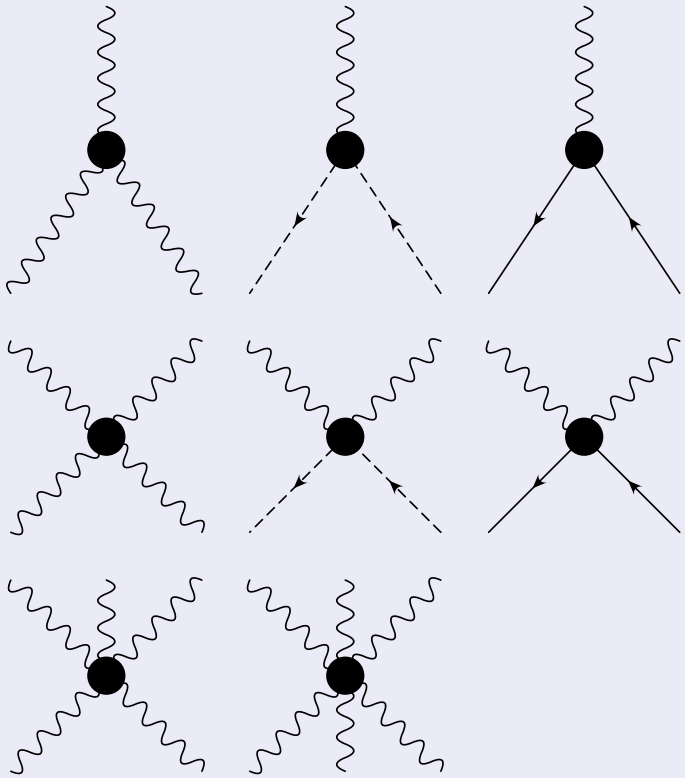
connected propagators = 1PI propagators

gluon

quark

ghost

vertices: 3-,4-,5-point functions and more



Ghosts

- Can appear in quantization procedure (Faddeev-Popov)
- Virtual particles (never occur in results for observables)
- Violate spin-statistics relation:
  spin 0, but anticommuting, "fermions"

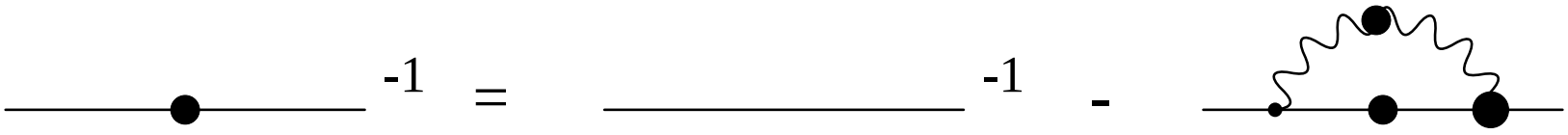
Some gauges feature ghosts: Landau gauge, maximal Abelian gauge

Ghost-free: e.g. Laplacian gauge

Example: Quark Propagator DSE

A DSE describes how particles propagate/interact.

Example: Quark propagator DSE, known as gap equation of QCD



Dynamical chiral symmetry breaking:

constituent quark mass is higher than current quark mass

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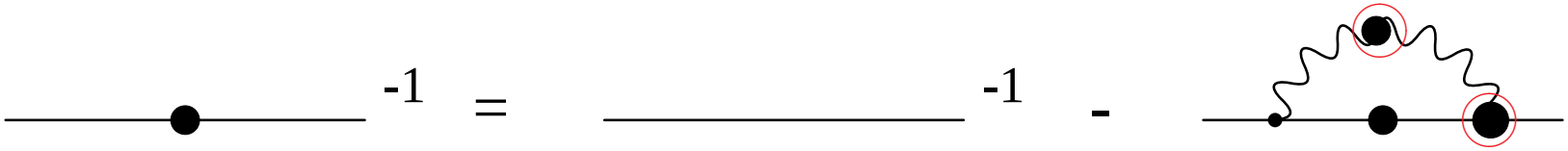

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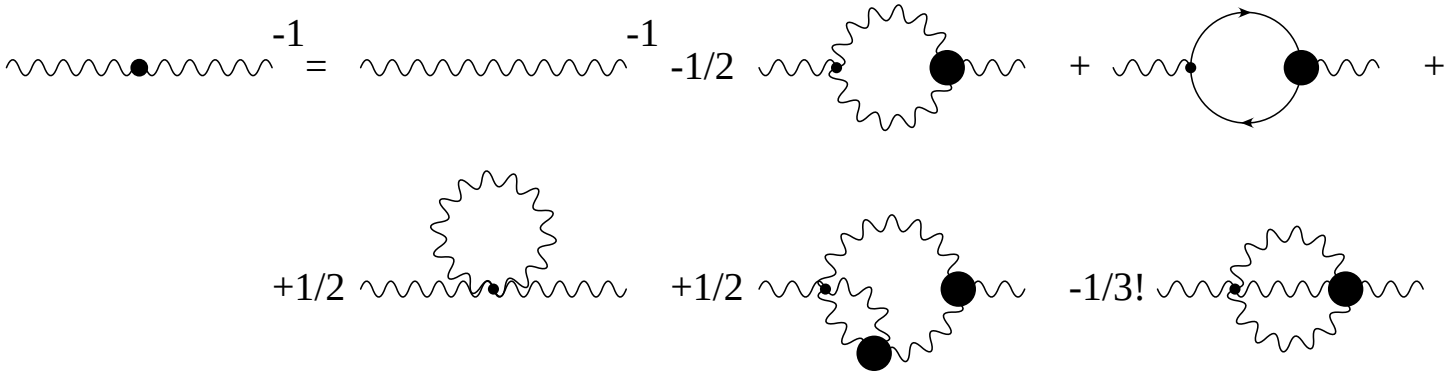
The Tower of DSEs

More interactions → more involved equations



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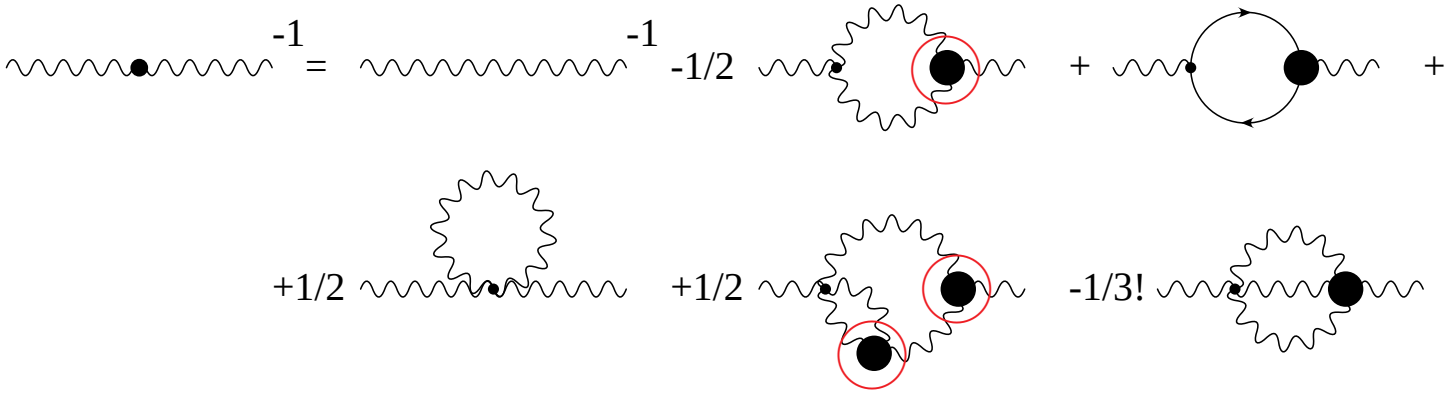

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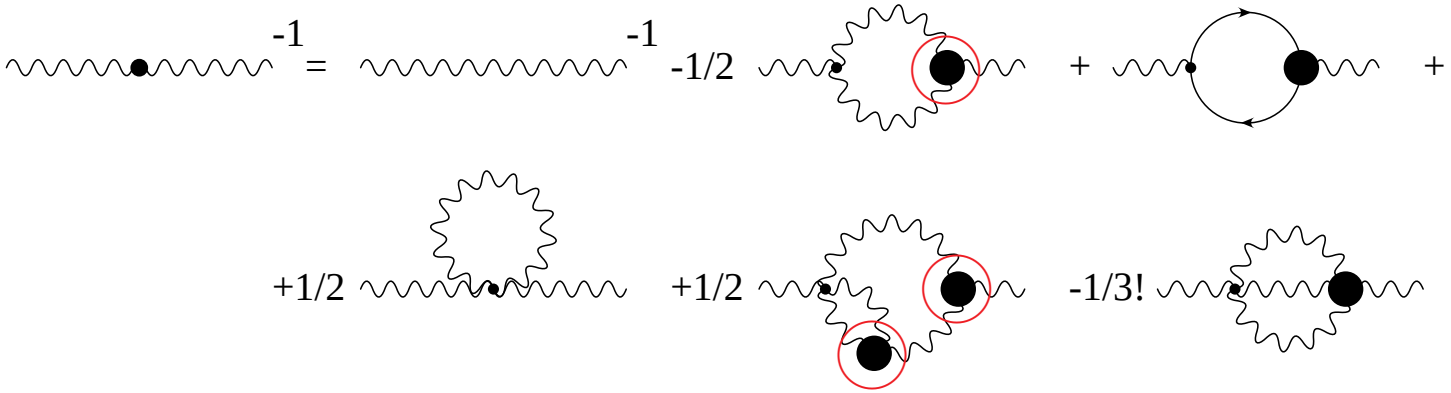

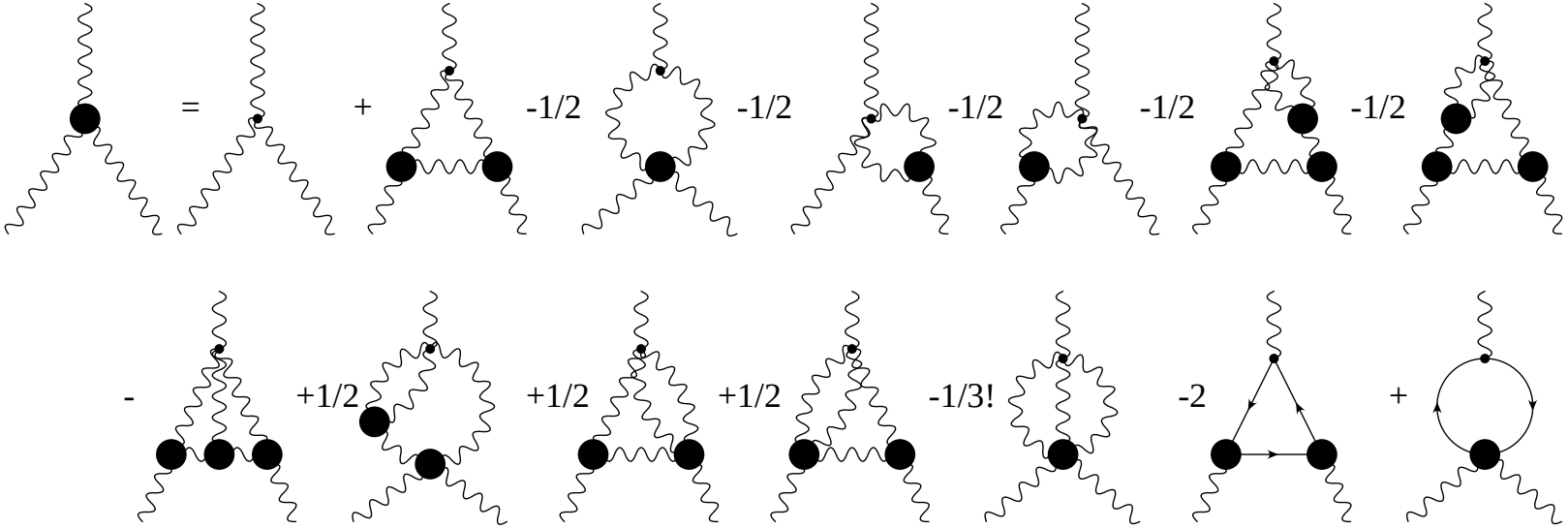

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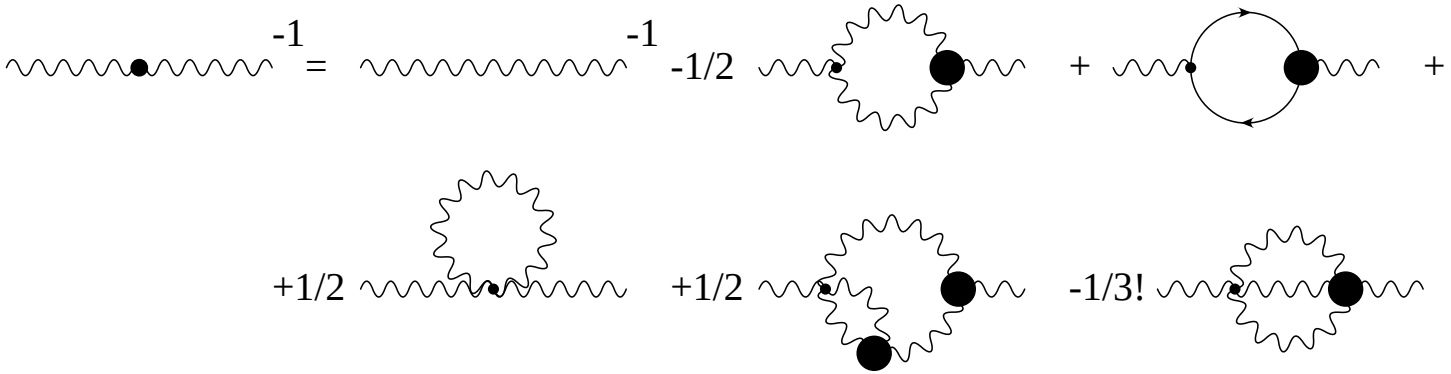
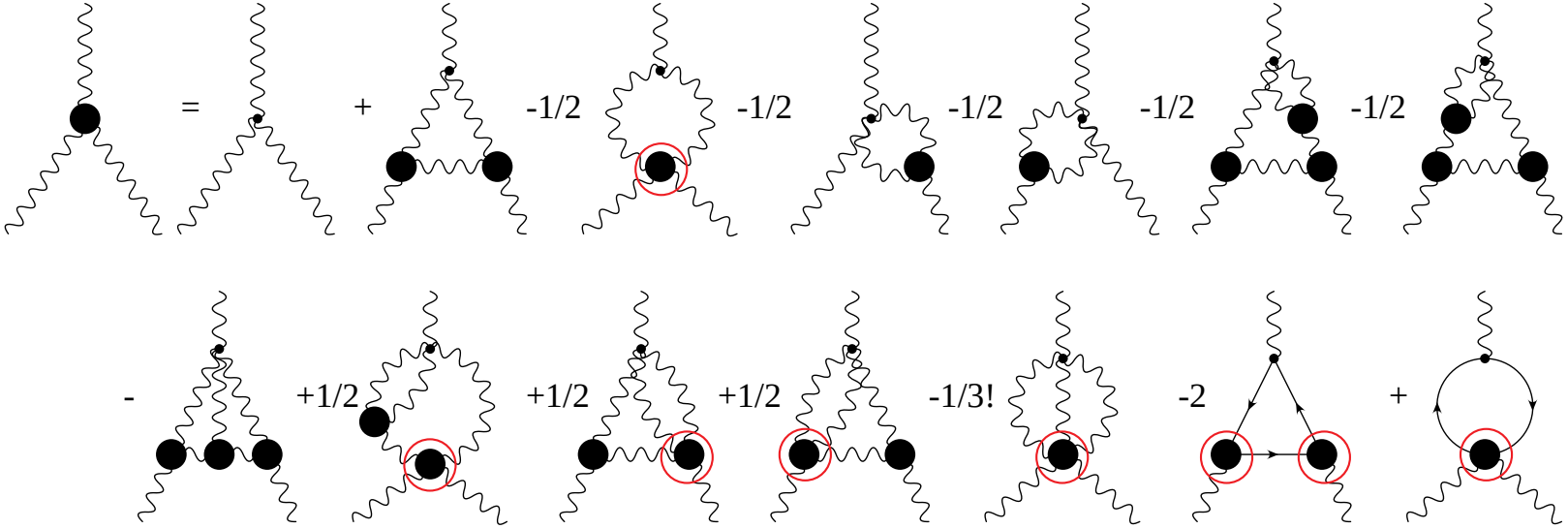
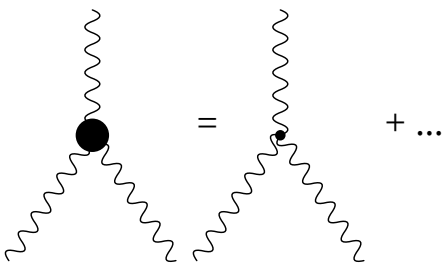




Structure

DSEs for a primitively divergent n-point function always have the following form:

- The n-point function itself appears bare and **1PI**.



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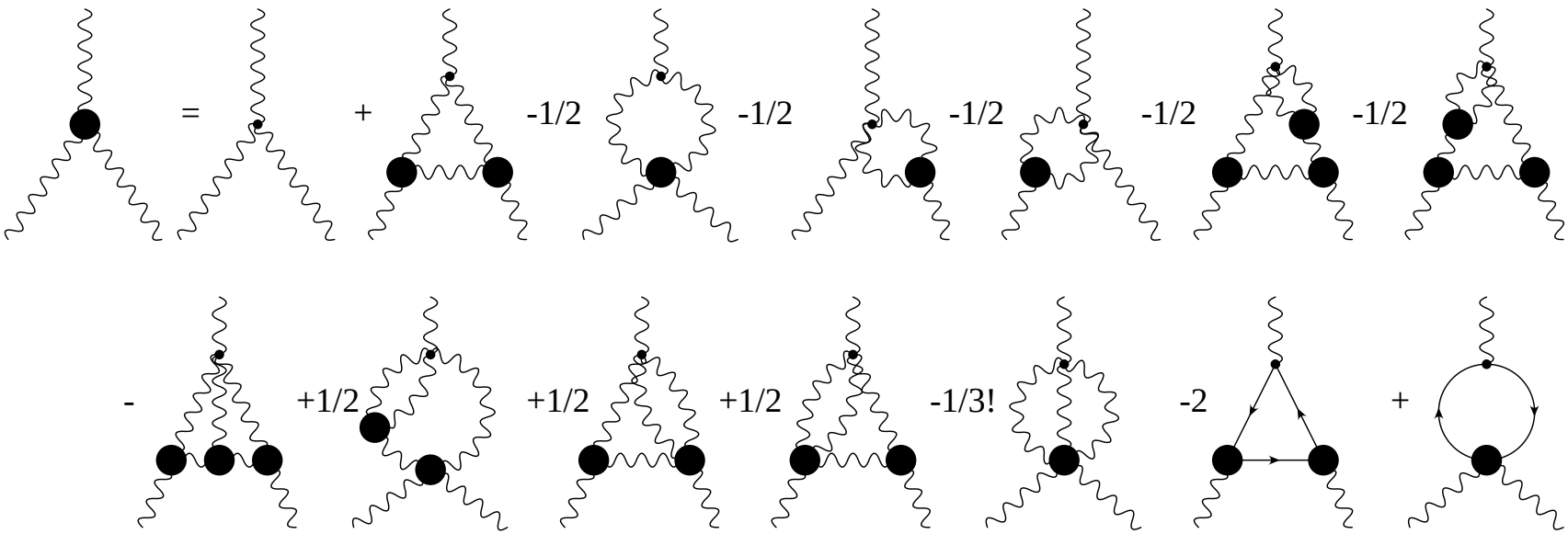
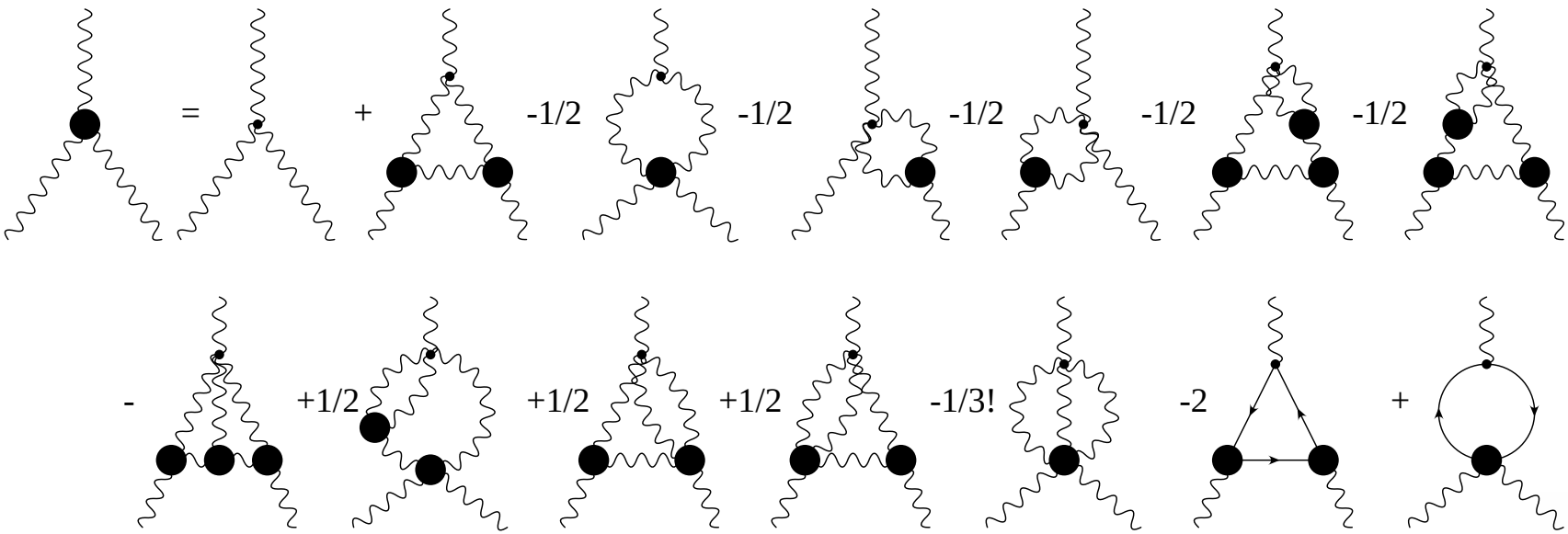


Structure

DSEs for a primitively divergent n-point function always have the following form:

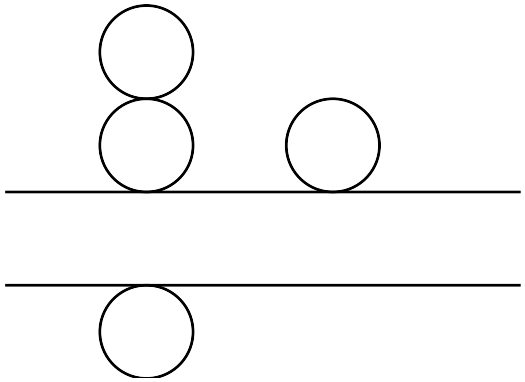
- The n-point function itself appears bare and **1PI**.
- All other diagrams contain exactly one bare vertex.
- There are always m-point functions with $m > n$.



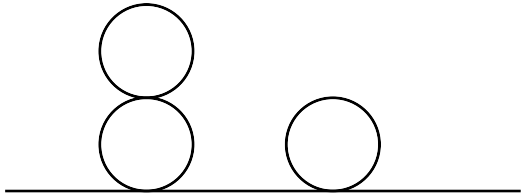
Generating Functionals

Generate the n-point functions of a theory by differentiation:
full, **connected**, **one-particle irreducible (1PI)**

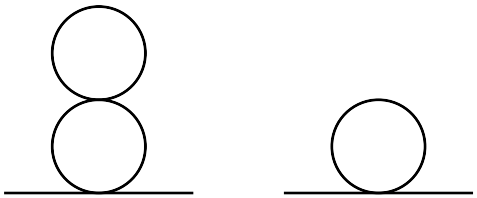
$$Z[J] = \sum_m \frac{1}{m!} G_m^{(f)} J^m$$



$$W[J] = \sum_m \frac{1}{m!} G_m^{(c)} J^m$$



$$\Gamma[A] = \sum_m \frac{1}{m!} G_m A^m$$



Relations:

$$Z[J] = e^{W[J]}$$
$$= \int D[A] e^{-S + A_i J_i}$$

$$\Gamma[A] = -W[J] + A_i J_i$$

The Total Derivative

Integral of a total derivative vanishes
(given appropriate boundary conditions):

$$\frac{\delta Z}{\delta A_i} = \int D[A] \frac{\delta}{\delta A_i} e^{-S + A_i J_i} =$$

$$= \left(- \frac{\delta S}{\delta A_i'} \bigg|_{A_i' = \delta/\delta J_i} + J_i \right) Z[J] = 0$$

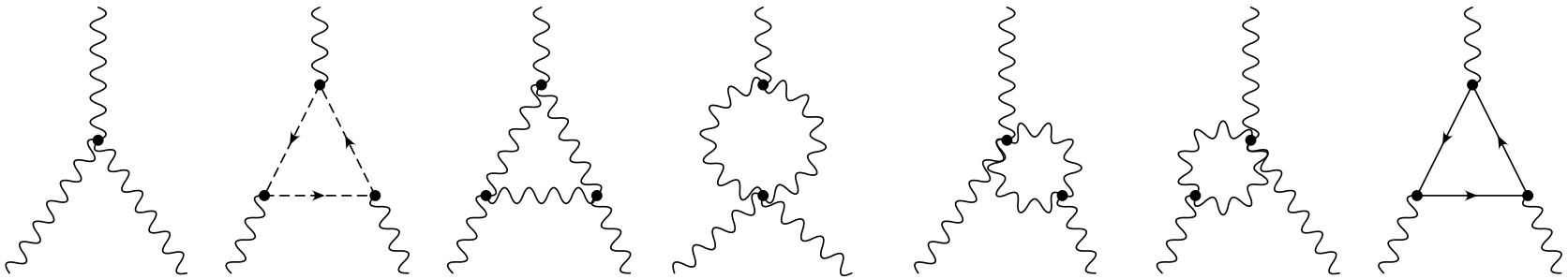
$$- \frac{\delta S}{\delta A_i'} \bigg|_{A_i' = \delta W/\delta J_i + \delta/\delta J_i} + J_i = 0$$ **connected n-point functions**

$$- \frac{\delta S}{\delta A_i'} \bigg|_{A_i' = A_i + D_{ij}^{(c)} \delta/\delta A_j} + \frac{\delta \Gamma}{\delta A_i} = 0$$ **1PI n-point functions**

Ultraviolet

Iteration of DSEs gives perturbation series.

Example: 3-gluon vertex up to one-loop



Truncations

- Infinite tower of equations $\longrightarrow$ Truncations needed
- Neglect higher n-point functions
- Better: make an ansatz that respects symmetries (Ward-Takahashi, Slavnov-Taylor)
- System can react very sensitive to truncations

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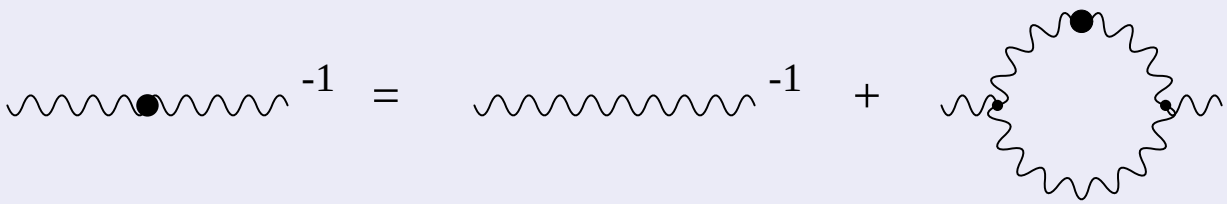
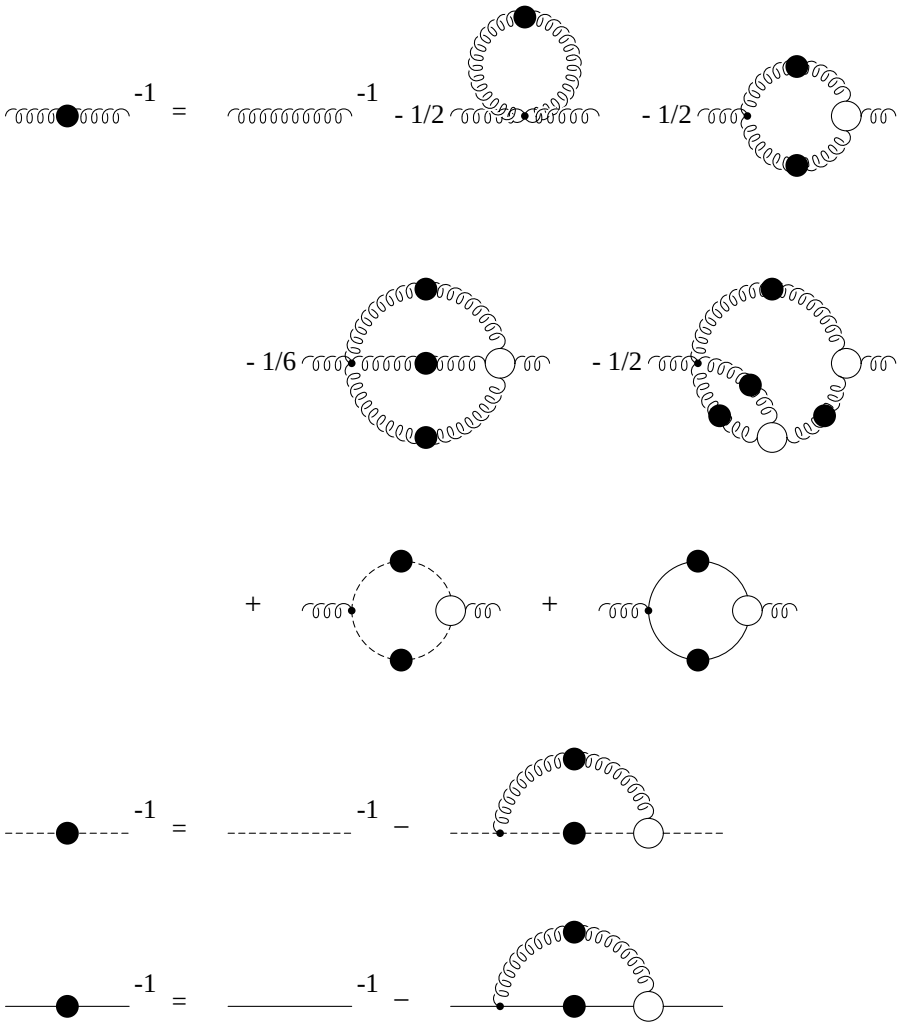
Example: Mandelstam approximation (1979)

Neglect all quark and ghost contributions:

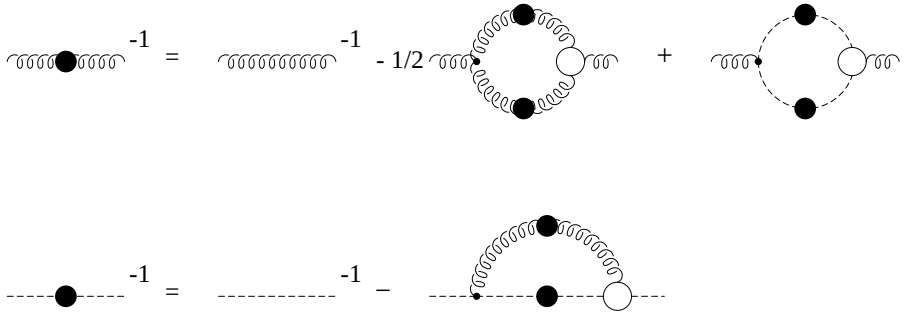


IR-divergent gluon propagator $\longrightarrow$ linear rising quark potential
$\longrightarrow$ confinement of quarks but WRONG picture

A More Complicated Example[1]



- Neglect quarks
- Neglect four-gluon vertices
- Restrain three-point vertices by Slavnov-Taylor identities



Allow for an analytic IR solution.

[1] von Smekal, Hauck, Alkofer, Ann. Phys. 267, 1

Infrared Behavior of Yang-Mills Theory

Gluon and **ghost** propagators:

$$D_{\mu\nu}(p^2) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{Z(p^2)}{p^2}, \quad D_G(p^2) = -\frac{G(p^2)}{p^2}$$

Analytic results in the IR for the dressing functions:

$$Z^{IR}(p^2) \sim (p^2)^{2\kappa}, \quad G^{IR}(p^2) \sim (p^2)^{-\kappa}$$

$\kappa = 0.59\ldots \quad \rightarrow$ **gluon** vanishing: $\sim (p^2)^{0.2}$
ghost divergent: $\sim (p^2)^{-1.6}$

Gluons cannot propagate over long distances $\rightarrow$ **gluons** confined

Renormalization group: confirms this behavior
Lattice: agreement except finite volume effects?

Gribov-Zwanziger and Kugo-Ojima scenarios

Propagators

- IR behavior can be determined from the ghost DSE:

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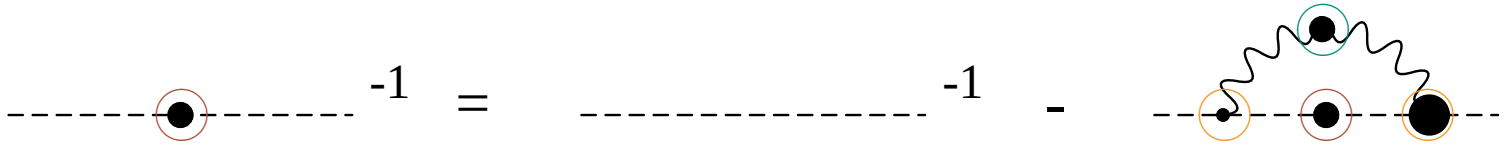

- Use bare ghost-gluon vertex
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$$\left(\frac{B\cdot(p^2)^\beta}{p^2}\right)^{-1} \sim \int \frac{d^4q}{(2\pi)^4} P_{\mu\nu} \frac{A\cdot(q^2)^\alpha}{q^2} \frac{B\cdot((p-q)^2)^\beta}{(p-q)^2} (p-q)_\mu q_\nu$$

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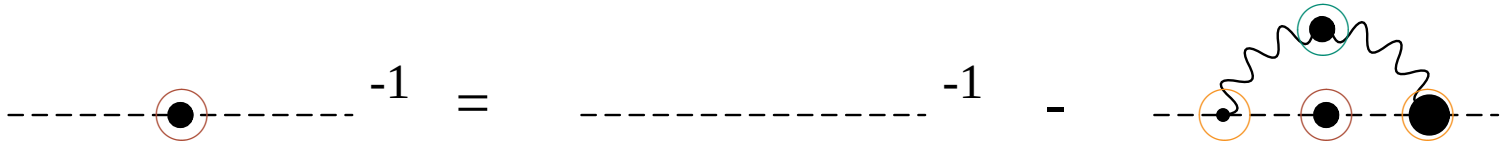

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- Only one momentum scale
  $\longrightarrow$ simple power counting is possible

$$1-\beta = 2+\alpha-1+\beta-1+\frac{1}{2}+\frac{1}{2} \longrightarrow -2\beta = \alpha$$

Power Counting

convention: $\beta = -\kappa$, IR exponent of ghost propagator

κ is determined from consistency relations between the gluon and ghost DSEs

Infrared exponents:
determine the IR behavior of the **dressing functions**

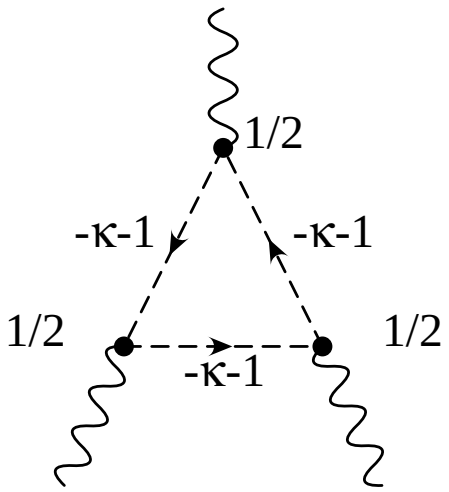
- gluon propagator: $(p^2)^{2\kappa}$
- ghost propagator: $(p^2)^{-\kappa}$
- Vertices: extract one **momentum scale**

$$H(p_1^2, p_2^2, p_3^2, \ldots) \to (p_1^2)^{\delta} H'\left(\frac{p_2^2}{p_1^2}, \frac{p_3^2}{p_1^2}, \ldots; \delta\right)$$

Skeleton Expansion

Ghost-triangle (three-gluon vertex)

$$(p^2)^{3(-\kappa-1)+2\frac{1}{2}+\frac{d}{2}} = (p^2)^{-3\kappa}$$



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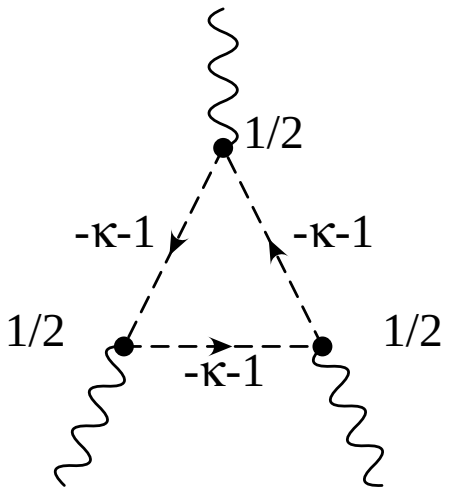
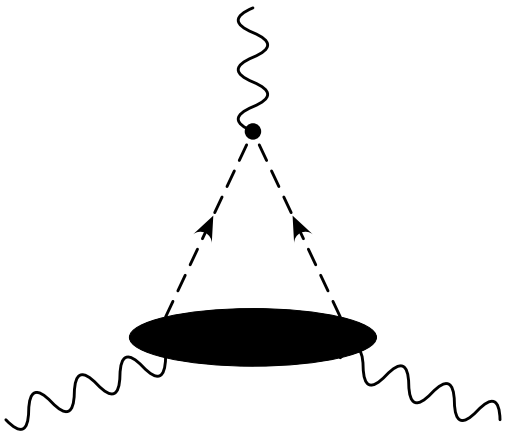


What about unknown vertices?



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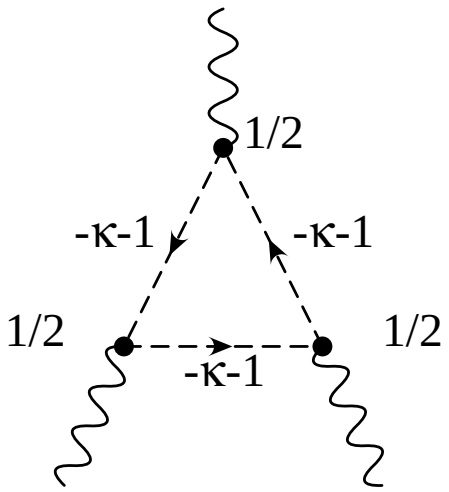
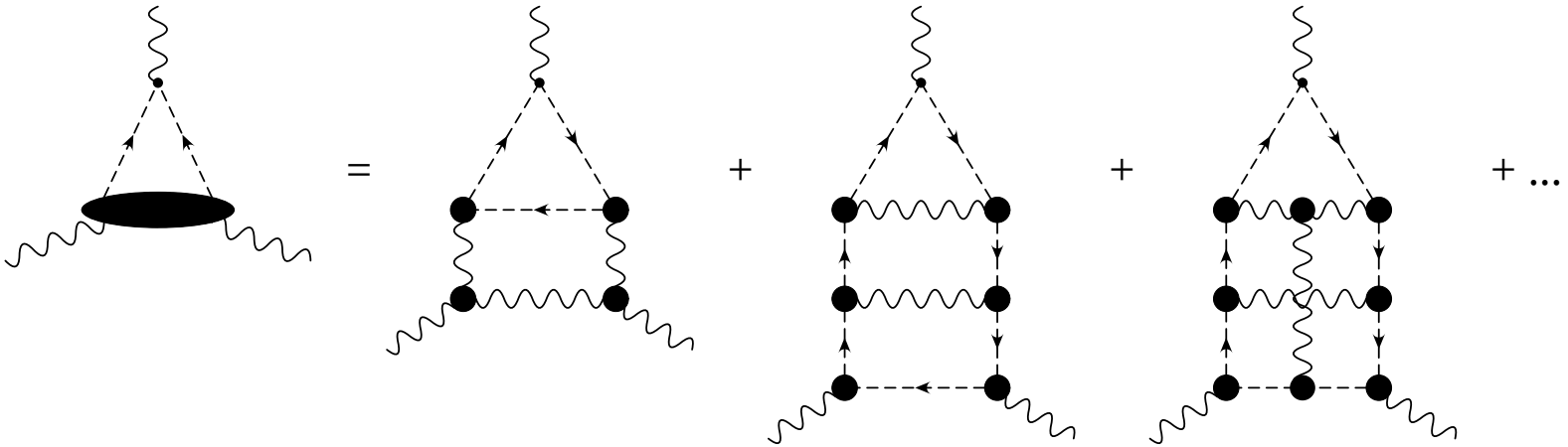


Employ skeleton expansion
$\sim$ loop expansion with dressed quantities



Skeleton Expansion

- All orders have the same IR exponent.
- Power counting for arbitrary number of internal vertices and propagators possible.
- IR exponent for vertices
  with $2n$ external ghosts and m gluons in d dimensions:

$$\delta_{2n,m} = (n-m)\kappa + (1-n)(\frac{d}{2} - 2)$$

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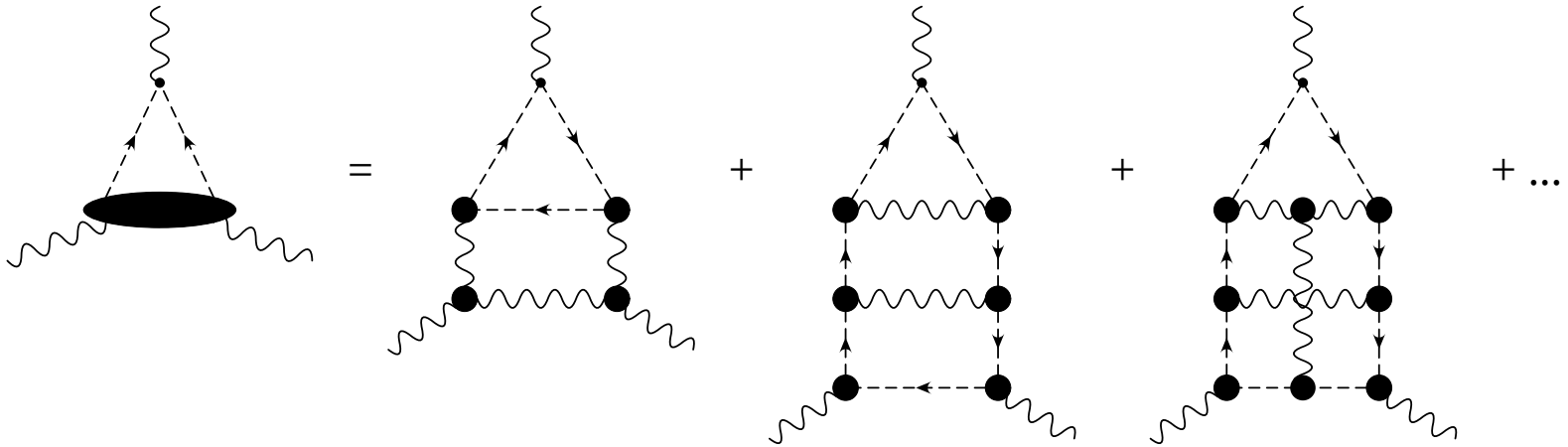

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Using the skeleton expansion
the IR behavior is determined without truncations.

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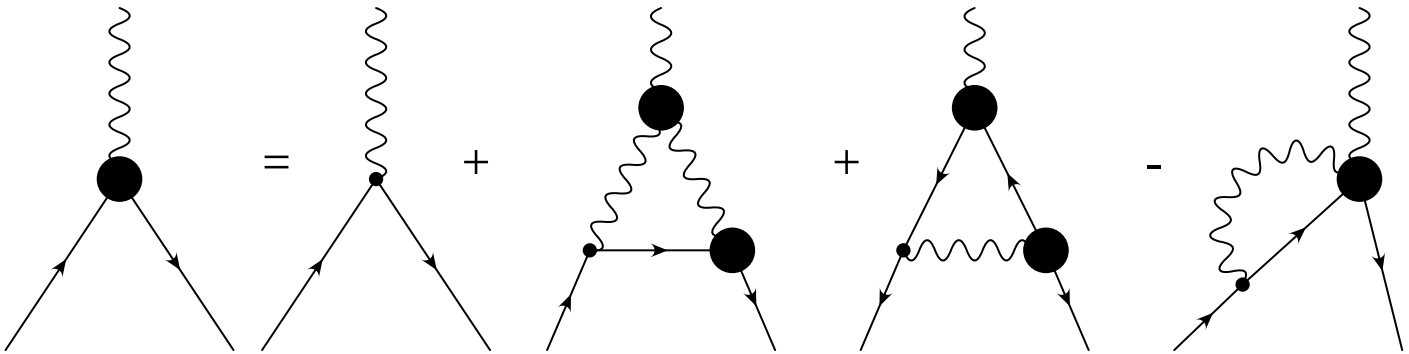

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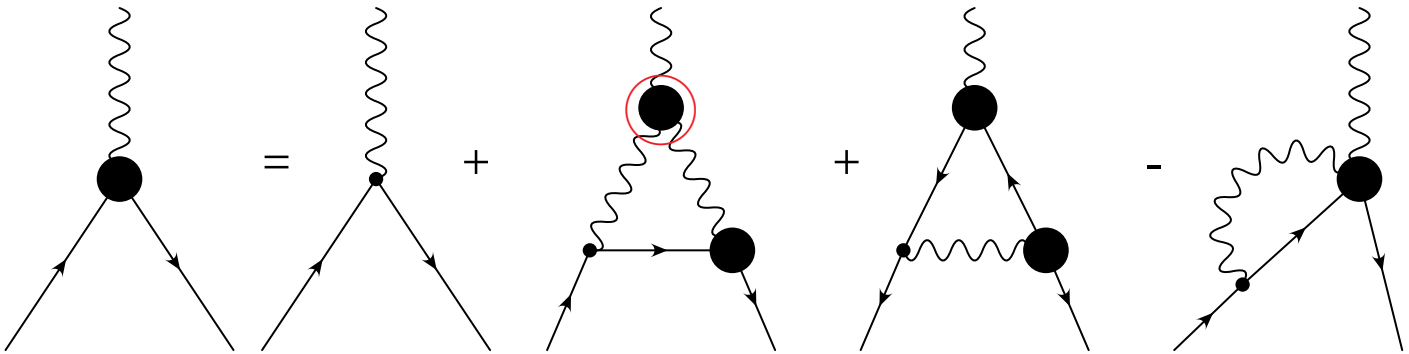

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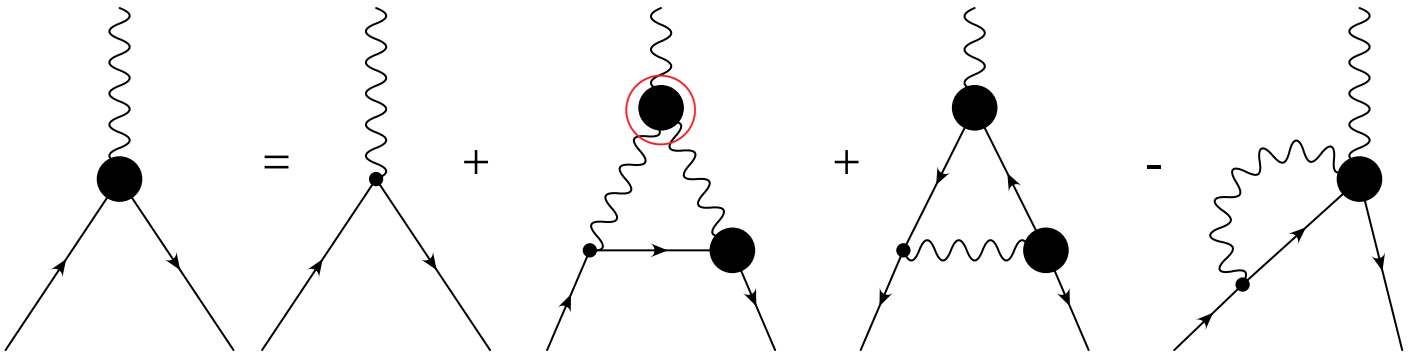
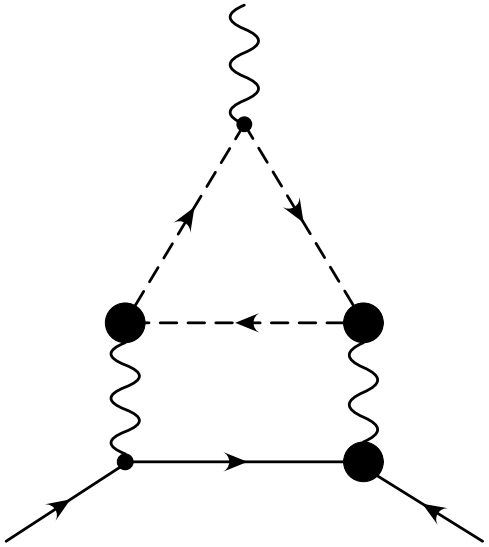


→ influence of ghosts in the quark-gluon DSE



Applications

QCD:

- Renormalizability
- Non-perturbative effects
  - IR behavior of propagators and vertex functions $\longrightarrow$ confinement
  - Dynamical chiral symmetry breaking
- Study of hadrons as composites of dressed quarks and gluons: Mass spectra, decay constants, electromagnetic form factors $\longrightarrow$ talks by Gernot Eichmann and Klaus Lichtenegger
- ...

Other field theories:

- Top quark condensate
- Condensed matter physics
- Calculation of the anomalous magnetic moment of the photon (Selym Villalba-Chavez)
- ...

Summary

- Non-perturbative continuum method
- Technically involved
- DSEs describe important aspects of QCD and hadron physics