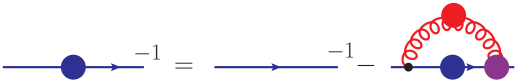
Integral equations

Markus Q. Huber



ECT* Doctoral Training Program
“Hadron physics with functional methods”

Trento, Italy

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JUSTUS-LIEBIG-UNIVERSITÄT GIESSEN

STRONG-2020

What is an integral equation?

Equation where an unknown function appears under an integral.

Example:

$$f(x) = \int_a^b K(x,t) g(t) dt$$

$f(x)$: known function
$g(x)$: unknown function $\rightarrow$ Solve for that.
$K(x,t)$: kernel

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Relation to differential equations.

Maxwell equations: Differential vs. integral form

Example: $\vec{\nabla} \cdot \vec{B} = 0$ $\int d\vec{A} \cdot \vec{B} = 0$

Classification of integral equations I

$f(x)$ known function
$g(t)$ unknown function

Fredholm integral equations

$$f(x) = \int_a^b K(x,t)g(t)dt$$

integration boundaries constant

Volterra integral equations

$$f(x) = \int_a^{x} K(x,t)g(t)dt$$

integration boundaries depend on x

Classification of integral equations II

$f(x)$ known function
$g(x)$ unknown function

Integral equations of **first kind**

$$f(x) = \int_a^b K(x,t)g(t)dt$$

unknown function only in integrand

Integral equations of **second kind**

$$g(x) = f(x) + \int_a^b K(x,t)g(t)dt$$

unknown function appears outside of integrand

Classification of integral equations III

$f(x)$ known function
$g(x)$ unknown function

Homogeneous/inhomogeneous integral equations

$$g(x) = f(x) + \int_a^b K(x,t)g(t)dt$$

$f(x) = 0 \rightarrow$ **homogeneous,** $f(x) \neq 0 \rightarrow$ **inhomogenous**

Linear integral equations

Unknown function appears linearly.

The propagator DSEs of QCD

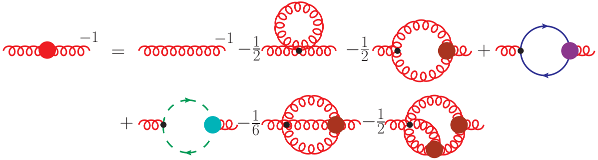
[MQH Phys. Rept. 879 (2020)]

Landau gauge

Most widely solved DSE: quark propagator DSE $\rightarrow$ See lectures Fischer, Maris, project 1.

Ghost propagator DSE: Same structure, but no Dirac traces, only one dressing $\rightarrow$ See project 5.

Gluon propagator DSE: One dressing, 2-loop diagrams, quadratic divergences $\rightarrow$ See project 5.



Renormalization

Most diagrams are UV divergent. $\rightarrow$ Renormalization needed.

UV divergence in renormalizable theory:

- Regularize integrals, here UV cutoff.
- Include counter terms in action to remove UV divergences.
- Renormalization conditions required.

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Particular renormalization scheme: MOM (momentum subtraction)

Ghost propagator DSE

$$G^{-1}(p^2) = \widetilde{Z}_3 - \Sigma(p^2)$$

Subtraction of DSE from itself at p_G^2:

$$G^{-1}(p^2) = G^{-1}(p_G^2) - \Sigma(p^2) + \Sigma(p_G^2)$$

Renormalization condition: $G(p_G^2)$ $\widetilde{Z}_3 = G^{-1}(p^2) + \Sigma(p^2)$

Renormalization of the quark propagator DSE

$$S(p)^{-1} = i\not{p}A(p^2) + B(p^2)$$

DSE:

Projection onto $A(p^2)$ and $B(p^2)$:

$$A(p^2) = Z_2 - \Sigma_A(p^2),$$
$$B(p^2) = Z_2\, Z_m\, m - \Sigma_B(p^2)$$

Renormalization conditions: $A(\mu^2) = 1, M(\mu^2) = B(\mu^2)/A(\mu^2) = m$

Bethe-Salpeter equation I

[More details: Eichmann, Sanchis-Alepuz, Williams, Alkofer, Fischer, Prog. Part. Nucl. Phys. 1 (2016)]

Idea: Bound state of 2 particles is encoded as a pole in the 4-point function.

Generic structure of a 4-point function G: trivial part + interacting part

$$G = G_0 + G_0 T G_0$$

G_0: product of two disconnected propagators
T: scattering matrix (amputated and connected part of G)
K: four-particle scattering kernel, two-particle irreducible

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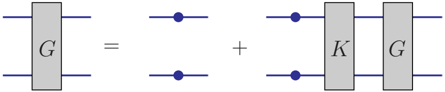
Dyson equation:

$$G = G_0 + G_0 K G$$



Resummation of
$G = G_0 + G_0 K G_0 + G_0 K G_0 K G_0 + \ldots$

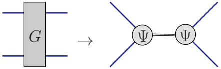
Nonperturbative!

Bethe-Salpeter equation II

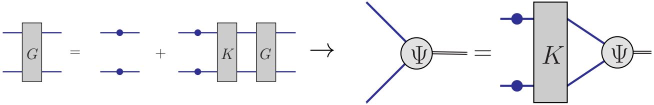
4-point functions at pole M^2:

$$G \rightarrow \frac{\Psi\,\bar{\Psi}}{P^2 + M^2}$$

Ψ: Wave function



Dyson equation at pole:



$$\Psi = G_0 K \Psi$$

General, e.g., M^2 could be complex $\rightarrow$ resonances.

Bethe-Salpeter equation II

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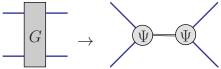
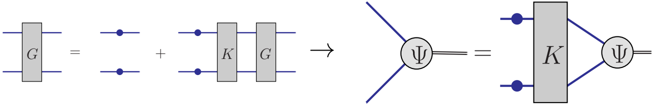


Dyson equation at pole:



$$\Psi = G_0 K \Psi$$

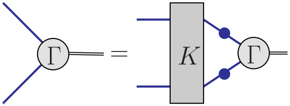
General, e.g., M^2 could be complex $\rightarrow$ resonances.

Same procedure for scattering matrix T:

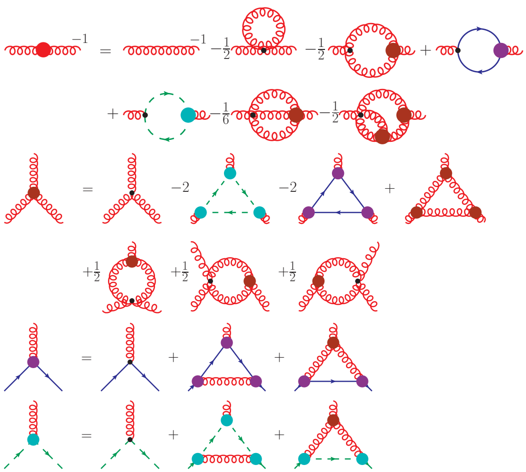
$$T \rightarrow \frac{\Gamma\,\bar{\Gamma}}{P^2 + M^2}, \quad \Rightarrow \quad \Gamma = K G_0 \Gamma$$

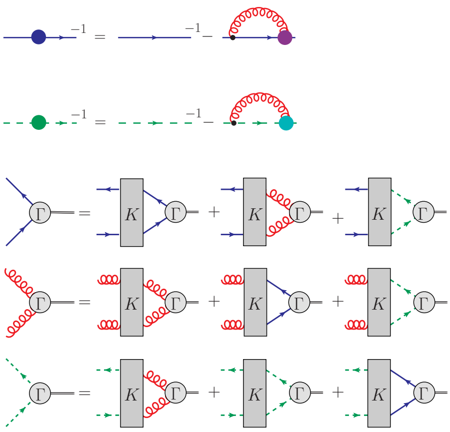
$$(\Psi = G_0 \Gamma)$$

Γ: Bethe-Salpeter amplitude



How to solve DSEs and BSEs?





Fixed point iteration

We consider the general case where a numeric solution is necessary.
(In some approximations exact solutions are possible.)

Self-consistent solution!

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Self-consistent solution!

Simplest method: Fixed point iteration

1. Choose a starting guess.
2. Calculate the right-hand side: integrals, (renormalization)
3. Use the right-hand side to determine a new solution.
4. Go back to 2 until convergence is reached.

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- Convergence: Stable solutions where nothing “changes” anymore, e.g.,
  $\sum_i |A_{\text{new}}(x_i) - A_{\text{old}}(x_i)| < \epsilon$
- Relaxation: Mix old and new solutions. Can help with convergence.

Coupled equations

Meta-iteration: Fixed point iterations for single equations.

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Example: Equations A, B, C, D

- Super-meta-iteration:
  - Iterate eq. A five times (or until converged).
  - Meta-iteration of subsystem B, C:
    - Iterate eq. B until converged.
    - Iterate eq. C once.
    - Check convergence of B, C.
  - Iterate eq. D until converged.
  - Check overall convergence for A, B, C, D.

Coupled equations

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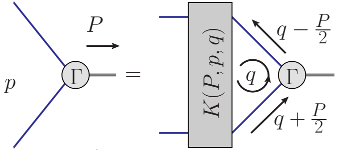
  - Iterate eq. D until converged.
  - Check overall convergence for A, B, C, D.

- Simple to realize.
- Easy to monitor progress.
- If we are lucky, it works. (We are lucky often.)

Ingredients of a BSE I



$$\Gamma(P;p) = \int \frac{d^4q}{(2\pi)^4} K(P,p,q) G_0(P;q) \Gamma(P;q)$$

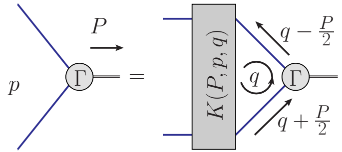
P: total momentum, center of mass momentum
p: relative momentum

Kernel $K(P, p, q)$

2-particle irreducible with respect to constituents, viz., does not contain diagrams generated by iteration.

Expression chosen by you. $\leftrightarrow$ Truncation.

Ingredients of a BSE II



- Decomposition of Bethe-Salpeter amplitude into Lorentz-invariant functions $f_i(p^2, p\cdot P, P^2)$:

$$\Gamma(P;p) = \sum_i f_i(p^2, p\cdot P, P^2)\boldsymbol{\tau_i(p, P)}\,.$$

  Choice of τ_i is problem dependent, e.g., $\tau_i \in \gamma_5 \cdot \{\mathbb{1}, \not{P}, \not{p}, [\not{p}, \not{P}]\}$ for pion.
  Equation needs to be projected onto the τ_i, see project 2.

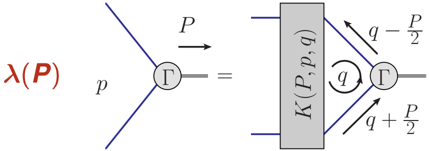
- Expansion of angle of $f_i(p^2, p\cdot P, P^2)$ in Chebyshev polynomials of the second kind:

$$f_i(p^2, p\cdot P, P^2) = \sum_{j=0}^{N} d_{i,j}(p^2, P^2)U_j(\cos\theta)$$

Solving a homogeneous BSE

Rewrite as eigenvalue problem

$$\boldsymbol{\lambda}(\boldsymbol{P})\,\Gamma(P) = \mathcal{K}\cdot\Gamma(P).$$



$\mathcal{K}$ is a matrix (discrete points)
$\lambda(P^2)=1$ is a solution to the BSE $\Rightarrow$ mass $M^2=-P^2$

$\rightarrow$ Vary P^2, find $\lambda(P^2)=1$, have mass $M^2=-P^2$.

Solving a homogeneous BSE

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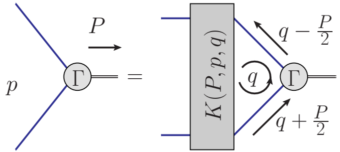
$\boldsymbol{\lambda(P)}$



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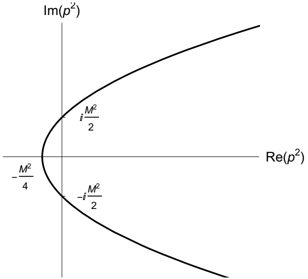
$\rightarrow$ Vary P^2, find $\lambda(P^2) = 1$, have mass $M^2 = -P^2$.

Calculation requires propagators for complex arguments:

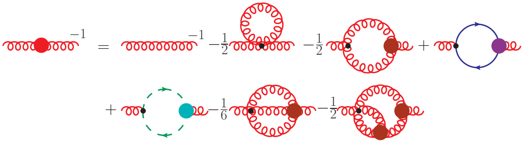
$$\left(q \pm \frac{P}{2}\right)^2 = \frac{P^2}{4} + q^2 \pm \sqrt{P^2\,q^2}\cos\theta$$
$$= -\frac{M^2}{4} + q^2 \pm \boldsymbol{i}\,M\sqrt{q^2}\cos\theta.$$



Minimization

Sometimes fixed point iteration just does not work...

Example: Gluon propagator
(depending on truncation)



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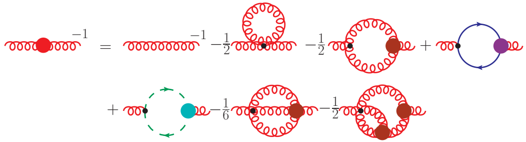

Idea: Expand dressings in some functions, e.g., Chebyshev polynomials, and find coefficients that solve the equation self-consistently:

$$Z(p^2) = exp\left(\sum_i^{N-1} c_i T_i(t(p^2))\right)$$

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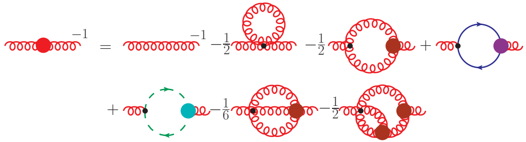

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Minimization problem

Example: $B(p^2) = m - \Sigma_B(p^2) \quad \rightarrow \quad \boldsymbol{E(p^2)} = m - \Sigma_B(p^2) - B(p^2) \overset{!}{=} \mathbf{0}$

Newton method

Minimize:

$$E^i = -Z^i(\{c\}) + DSE^i(\{c\})$$

Z^i: dressing function for external point x_i
$\{c\}$: coefficients of expansion of Z^i

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Newton method $d = 1$: Find zero of $f(x)$:

$$x_{\text{new}} = x_{\text{old}} - \frac{f(x_{\text{old}})}{f'(x_{\text{old}})}$$

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Generalized Newton method: $x \to c$, $f \to E$, $f' \to J$

$$c^i_{\text{new}} = c^i_{\text{old}} - \lambda \sum_k (J^{-1})^{ik} E^k$$

Jacobian $J^{ik} = \frac{\partial E^k}{\partial c^i}$

λ: Backtracking parameter to adjust step size.

Calculation of Jacobian

Could calculate derivative exactly, but approximation is sufficient: Broyden's method

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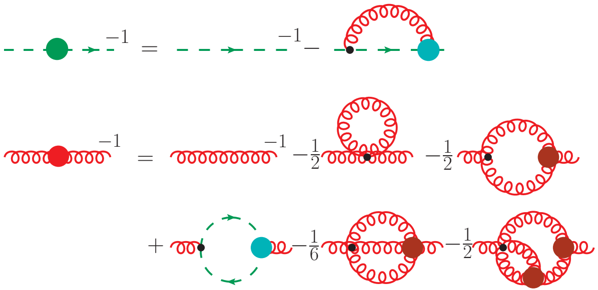
Forward derivative:

$$J^{ik} = \frac{E^k(c^i + h) - E^k(c^i)}{h}$$

$\rightarrow$ Calculate E once with the given coefficients ($E^k(c^i)$)
and then vary each coefficient by h ($E^k(c^i + h)$). E.g. $h = 10^{-3}$

Coupled system of equations

Example: Gluon and ghost propagators in Yang-Mills theory



Both equations depend on each other: $E^i \to \boldsymbol{E^{(i,a)}}$, where a labels the DSE.

$\rightarrow$ Jacobian has block-diagonal form.

Remarks on Newton method

Stopping criterion

Minimize $E^i \rightarrow$ Iterate and monitor $|E|$.

- Stop when defined threshold is reached.
- Converges quadratically fast close to the minimum.
- Can see if one gets stuck in local minimum, $|E| > 0$.

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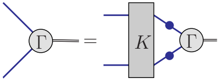
- Stop when defined threshold is reached.
- Converges quadratically fast close to the minimum.
- Can see if one gets stuck in local minimum, $|E| > 0$.

- Starting point can be important.
- Black box-like.
- Computationally more costly (calculation of J).
- Stable and reliable, once it works.
- Alternative to fixed point iteration when that fails.

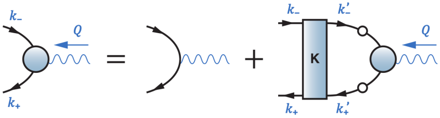
Use Newton when fixed point iteration fails.

Summary

- Eigenvalue problem: Homogeneous BSE



- Fixed point iteration: Simplest available method.
  Also for inhomogeneous BSE, see project 3, and equ. of motion of *n*PI effective actions.



- Minimization (Newton method): More powerful, more complicated.