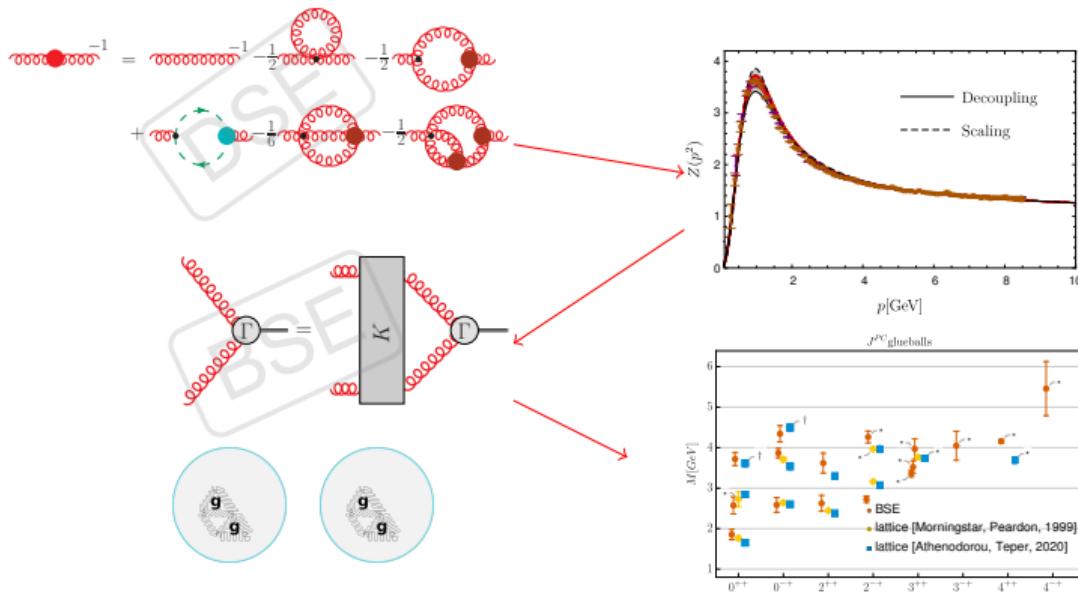


# The gluonic sector of QCD: From propagators to glueballs



JUSTUS-LIEBIG-  
**T** UNIVERSITÄT  
GIESSEN

**DFG** Deutsche  
Forschungsgemeinschaft

**STRONG**  
**2020**

Markus Q. Huber

Institute of Theoretical Physics  
Giessen University

Markus Q. Huber (Giessen University)

ECT\* Doctoral Training Program  
“Hadron physics with functional methods”

Trento, Italy, May 13, 2022

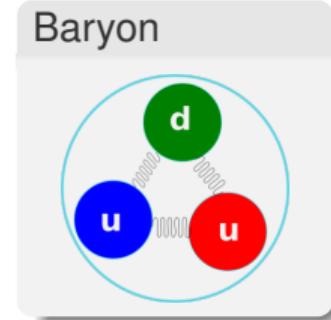
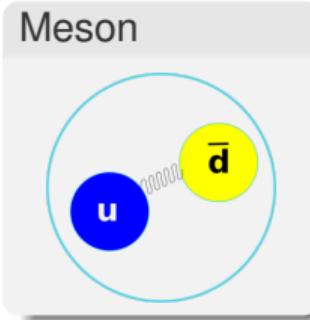
The gluonic sector of QCD

May 13, 2022

1/46

# Bound states of the strong interaction

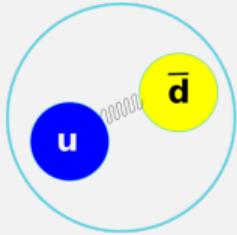
Quark model 1964: abundance  
of known states



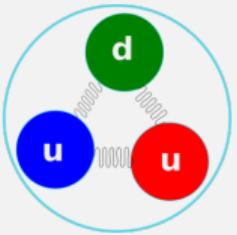
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Quark model 1964: abundance of known states

Meson

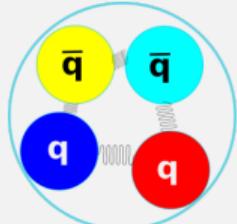


Baryon

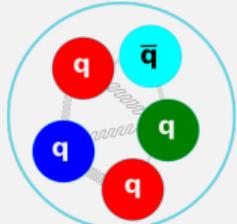


Exotics:

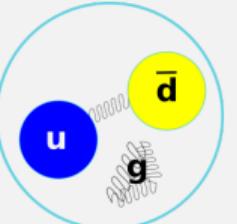
Tetraquark



Pentaquark



Hybrid



Glueball

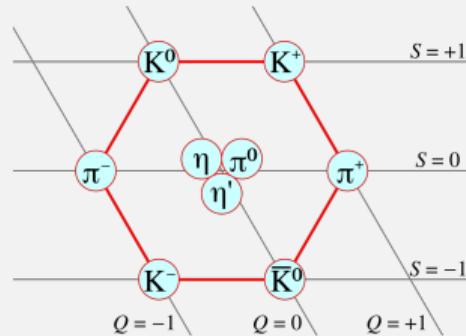


# Multiplets

Quark model

Classification in terms of mesons  
or baryons → multiplets

Outside this classification  
→ exotics



$$J^{PC} = 0^{-+}$$

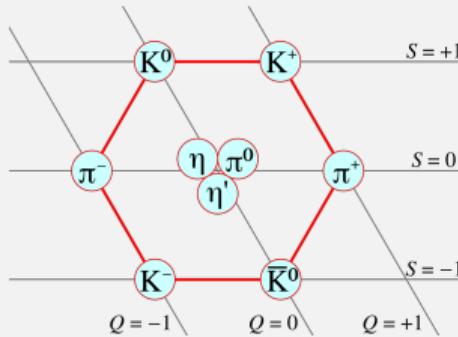
# Multiplets

Quark model

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Outside this classification  
→ exotics

Classification not always easy, e.g., scalar sector  $J^{PC} = 0^{++}$ :



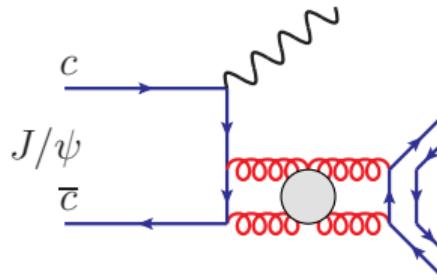
$$J^{PC} = 0^{-+}$$

glueball candidates

$f_0(500)$
$f_0(980)$
$f_0(1370)$
$f_0(1500)$
$f_0(1710)$

+ more states not considered established

# Glueballs from $J/\psi$ decay

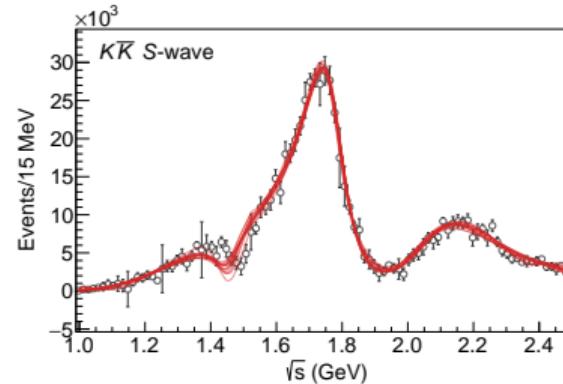
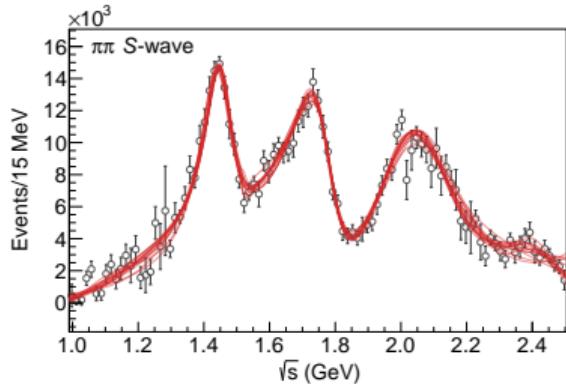


Coupled-channel analyses of exp. data (BESIII):

- +add. data, largest overlap with  $f_0(1770)$
- largest overlap with  $f_0(1710)$

[Sarantsev, Denisenko, Thoma, Klempt, Phys. Lett. B 816 (2021)]

[Rodas et al., Eur.Phys.J.C 82 (2022)]



# Glueball calculations: Lattice

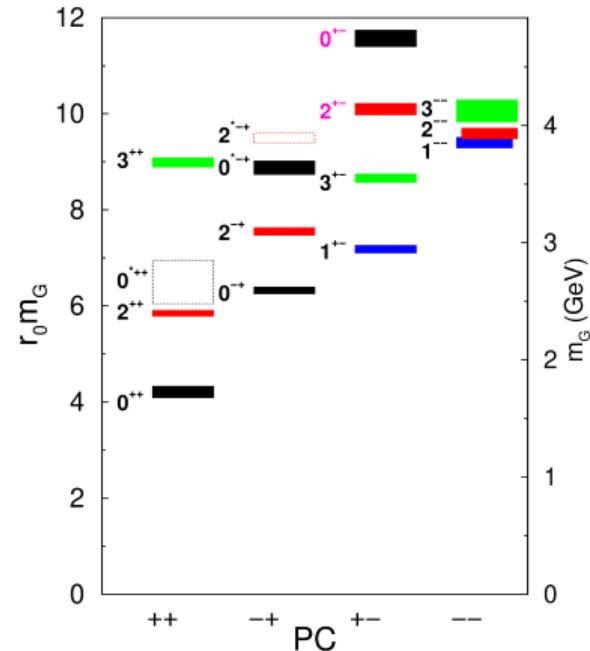
## Lattice methods

Pure gauge theory:

No dynamic quarks.

→ “Pure” glueballs

- [Morningstar, Peardon, Phys. Rev. D60 (1999)]: standard reference
- [Athenodorou, Teper, JHEP11 (2020)]: improved statistics, more states



[Morningstar, Peardon, Phys. Rev. D60 (1999)]

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“Real QCD”:

- [Gregory et al., JHEP10 (2012)]

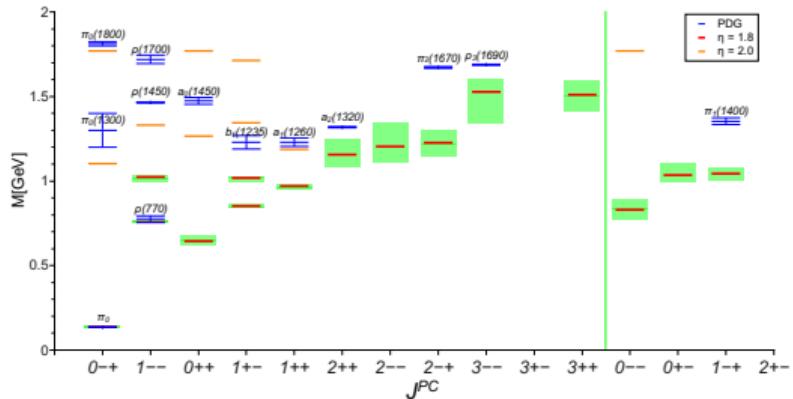
## Challenging:

- Much higher statistics required (poor signal-to-noise ratio)
- Continuum extrapolation and inclusion of fermionic operators still to be done
- Mixing with  $\bar{q}q$  challenging
- $m_\pi = 360 \text{ MeV}$
- Tiny (e.g.,  $0^{++}, 2^{++}$ ) to moderate unquenching effects (e.g.,  $0^{-+}$ ) found

No quantitative results yet.

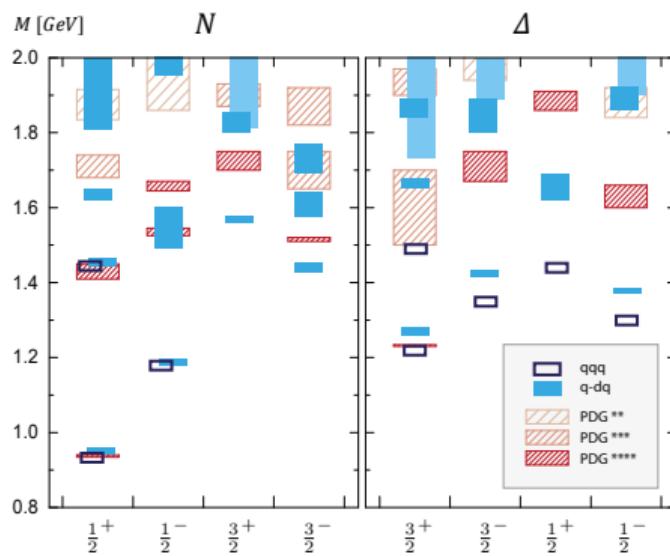
# Functional spectrum calculations

Functional methods successful in describing many aspects of the hadron spectrum qualitatively and quantitatively!



[Fischer, Kubrak, Williams, Eur.Phys.J.A50 (2014)]

Rainbow-ladder with Maris-Tandy (or similar) has been the workhorse for more than 20 years.



[Eichmann, Fischer, Sanchis-Alepuz, Phys.Rev.D94 (2016)]

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There is no rainbow for gluons!

Model based:

- [Meyers, Swanson, Phys.Rev.D87 (2013)]
- [Sanchis-Alepuz, Fischer, Kellermann, von Smekal, Phys.Rev.D92, (2015)]
- [Souza et al., Eur.Phys.J.A56 (2020)]
- [Kaptari, Kämpfer, Few Body Syst.61 (2020)]

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Calculation based:

- [MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C80 (2020)]
- [MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C81 (2021)]

Extreme sensitivity on input!

# Glueballs as bound states

Hadron masses from correlation functions of color singlet operators.

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Hadron masses from correlation functions of color singlet operators.

Example: For  $J^{PC} = 0^{++}$  glueball take  $O(x) = F_{\mu\nu}(x)F^{\mu\nu}(x)$ :

$$\mathcal{D}(\mathbf{x} - \mathbf{y}) = \langle O(x)O(y) \rangle$$

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Lattice: Mass exponential Euclidean time decay:

$$\lim_{t \rightarrow \infty} \langle O(x)O(0) \rangle \sim e^{-tM}$$

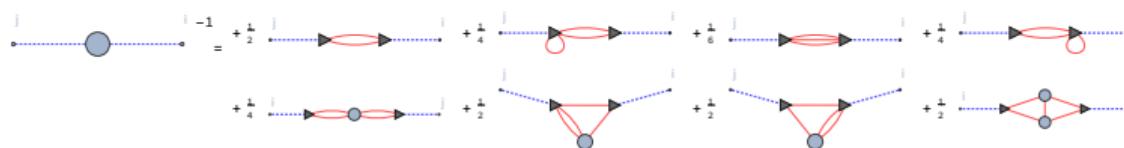
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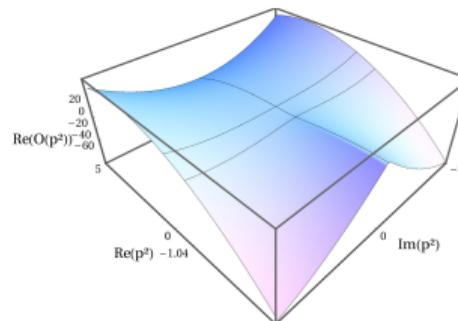
Functional approach: Complicated object in a diagrammatic language, 2-, 3- and 4-gluon contributions [MQH, Cyrol, Pawłowski, Comput.Phys.Commun. 248 (2020)]



+ 3-loop diagrams

Leading order:

[Windisch, MQH, Alkofer, Phys.Rev.D87 (2013)]



# Glueballs as bound states

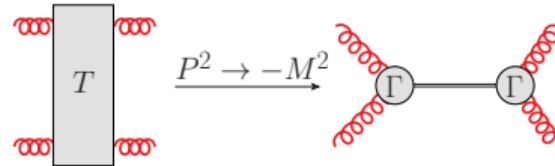
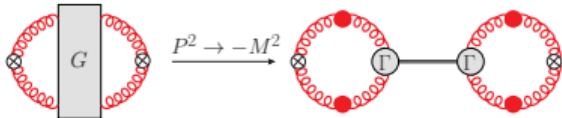
Hadron masses from correlation functions of **color singlet operators**.

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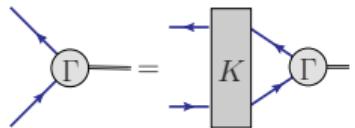
$$D(x - y) = \langle O(x)O(y) \rangle$$

Put total momentum **on-shell** and consider individual 2-, 3- and 4-gluon contributions. →  
Each can have a pole at the glueball mass.

$A^4$ -part of  $D(x - y)$ , total momentum on-shell:

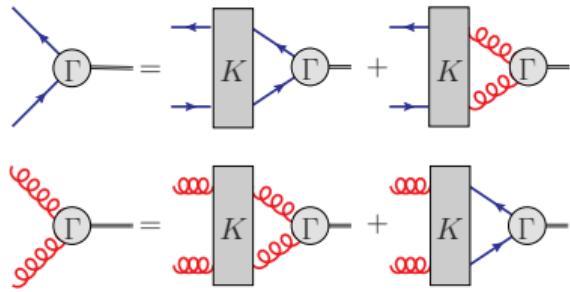


# Bound state equations for QCD



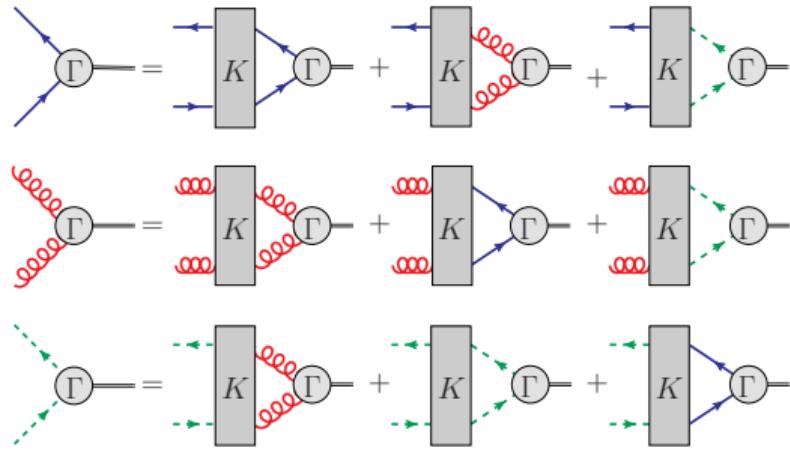
- Require scattering kernel  $K$  and propagator.

# Bound state equations for QCD



- Require scattering kernels  $K$  and propagators.
- Quantum numbers determine which amplitudes  $\Gamma$  couple.

# Bound state equations for QCD



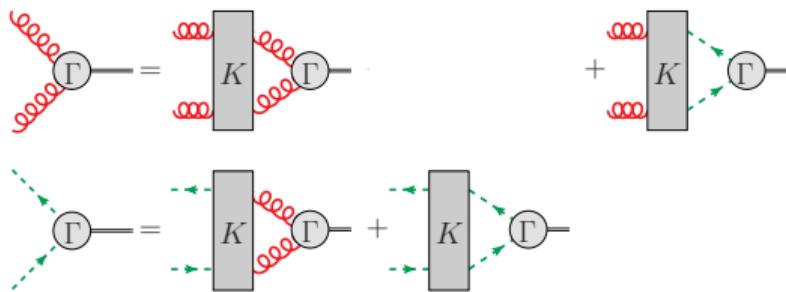
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- **Ghosts** from gauge fixing

## One framework

- Natural description of mixing.
- Similar equations for hadrons with more than two constituents

# Bound state equations for QCD

Focus on pure glueballs.



- Require scattering kernels  $K$  and propagators.
- Quantum numbers determine which amplitudes  $\Gamma$  couple.
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One framework

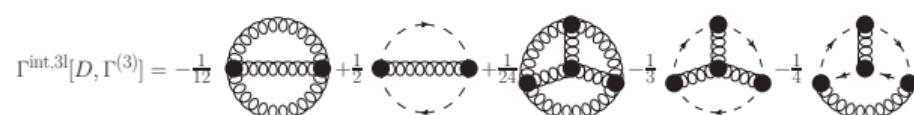
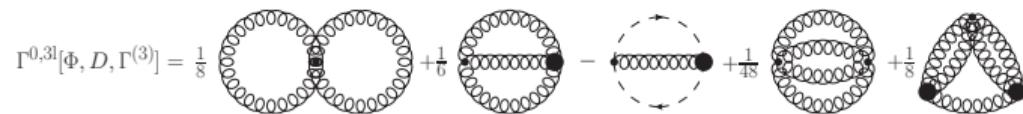
- Natural description of mixing.
- Similar equations for hadrons with more than two constituents

# Kernel construction

From 3PI effective action truncated to three-loops:

[Fukuda, Prog. Theor. Phys 78 (1987); McKay, Munczek, Phys. Rev. D 40 (1989); Sanchis-Alepuz, Williams, J. Phys: Conf. Ser. 631 (2015); MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C80 (2020)]

$$\Gamma^{3l}[\Phi, D, \Gamma^{(3)}] = \Gamma^{0,3l}[\Phi, D, \Gamma^{(3)}] + \Gamma^{\text{int},3l}[\Phi, D, \Gamma^{(3)}]$$



Kernels constructed by cutting two legs:  
gluon/gluon, ghost/gluon, gluon/ghost, ghost/ghost

# Kernels

Systematic derivation from 3PI effective action: Self-consistent treatment of 3-point functions requires 3-loop expansion.

$$K = \text{I} + \frac{1}{2} \text{X} - \text{X} + \frac{1}{2} \text{II} - \frac{1}{2} \text{III}$$

$$K = \text{I} + \frac{1}{2} \text{X} + \frac{1}{2} \text{X}$$

$$K = \text{I} + \frac{1}{2} \text{X}$$

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# Correlation functions and their equations of motion

Prototype: Equation of motion (Dyson-Schwinger equations) for the quark propagator



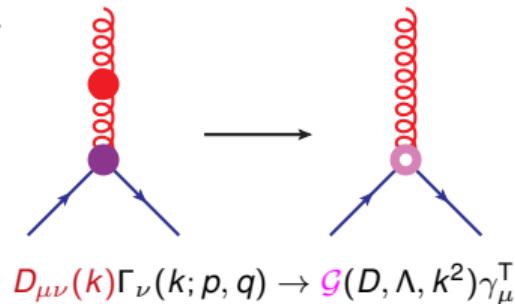
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Dealing with the unknowns:

Model



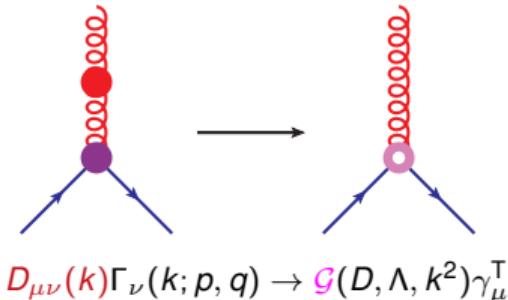
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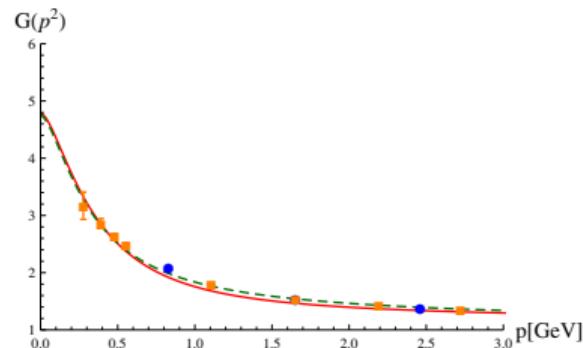
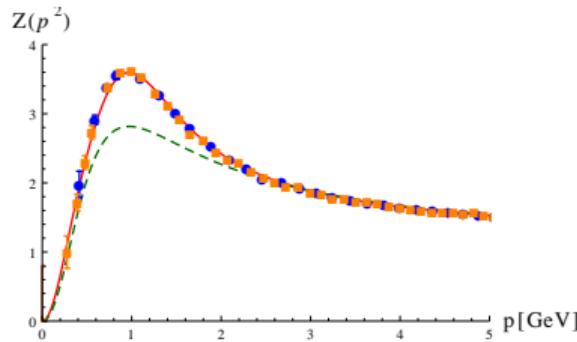


Calculate

$$\begin{aligned} \text{---}^{-1} &= \text{---}^{-1} - \frac{1}{2} \text{---}^{-1} \text{---} + \frac{1}{2} \text{---}^{-1} \text{---} + \text{---}^{-1} \\ &+ \text{---}^{-1} - \frac{1}{6} \text{---}^{-1} \text{---} + \frac{1}{2} \text{---}^{-1} \text{---} \\ = & \text{---} + \text{---} + \text{---} \end{aligned}$$

# The importance of self-consistency

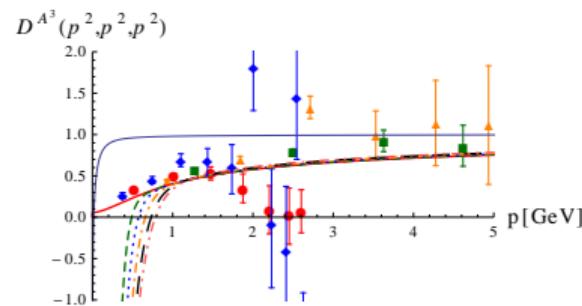
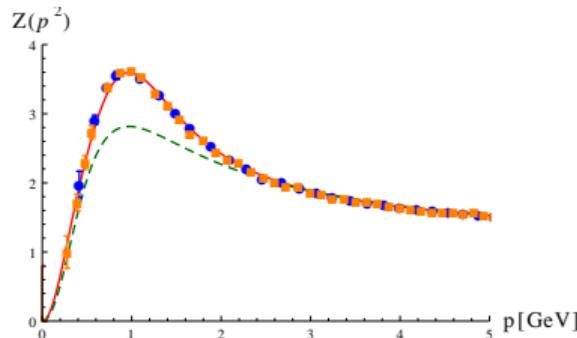
Propagators from modeled three-gluon vertex [MQH, von Smekal, JHEP 04 (2013)]:  
 Optimize parameters so that propagators match lattice results.



Agreement with lattice is not sufficient!

# The importance of self-consistency

Propagators from modeled three-gluon vertex [MQH, von Smekal, JHEP 04 (2013)]:  
 Optimize parameters so that propagators match lattice results.



Agreement with lattice is not sufficient!

- Vertex couplings show violations of gauge invariance.
- Model optimized for one quantity. → Restricted applicability to other quantities, e.g., glueballs.

# Correlation functions of quarks and gluons

Equations of motion: 3-loop 3PI effective action

→ [Review: MQH, Phys.Rept. 879 (2020)]

$$\text{Diagram}^{-1} = \text{Diagram}^{-1} - \frac{1}{2} \text{Diagram}^{-\frac{1}{2}} - \frac{1}{2} \text{Diagram}^{-\frac{1}{2}} + \text{Diagram}^0$$

$$+ \text{Diagram}^{-\frac{1}{6}} - \frac{1}{2} \text{Diagram}^{-\frac{1}{2}} - \frac{1}{2} \text{Diagram}^{-\frac{1}{2}}$$

$$\text{Diagram}^{-2} = \text{Diagram}^{-2} - \frac{1}{2} \text{Diagram}^{-1} - \frac{1}{2} \text{Diagram}^{-1} + \text{Diagram}^0$$

$$+ \frac{1}{2} \text{Diagram}^{+\frac{1}{2}} + \frac{1}{2} \text{Diagram}^{+\frac{1}{2}} + \frac{1}{2} \text{Diagram}^{+\frac{1}{2}}$$

$$\text{Diagram}^{-1} = \text{Diagram}^{-1} + \text{Diagram}^0 + \text{Diagram}^0$$

$$= \text{Diagram}^{-1} + \text{Diagram}^{-\frac{1}{2}} + \text{Diagram}^{-\frac{1}{2}}$$

$$\text{Diagram}^{-1} = \text{Diagram}^{-1} - \text{Diagram}^{-\frac{1}{2}}$$

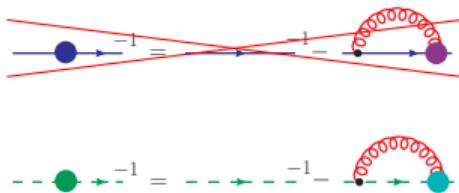
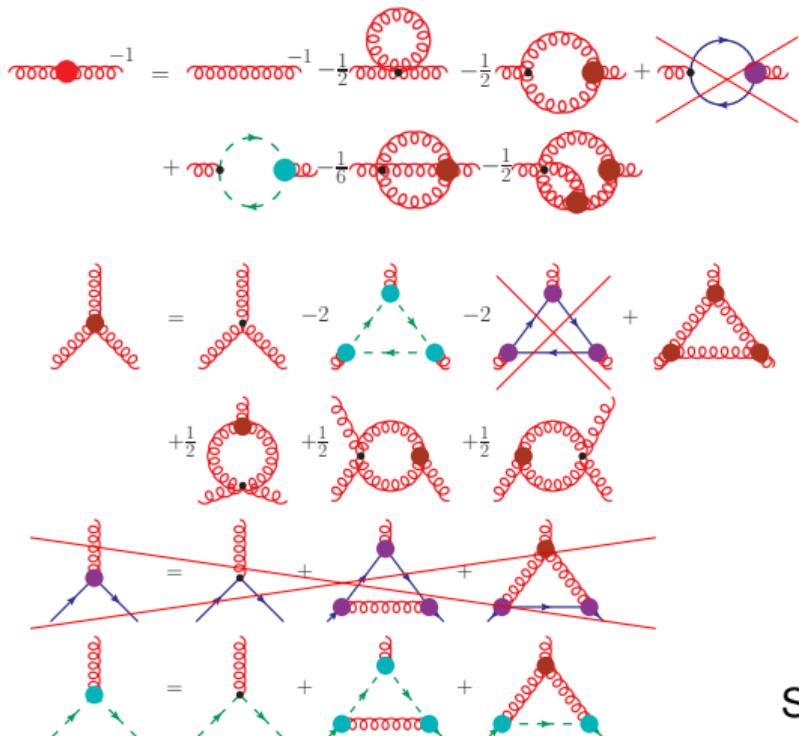
$$- \text{Diagram}^{-\frac{1}{2}} = \text{Diagram}^{-\frac{1}{2}} - \text{Diagram}^{-\frac{1}{2}}$$

- Conceptual and technical challenges: nonperturbative renormalization, two-loop diagrams, convergence, size of kernels, ...
- Self-contained: Only parameters are the **strong coupling and the quark masses!**

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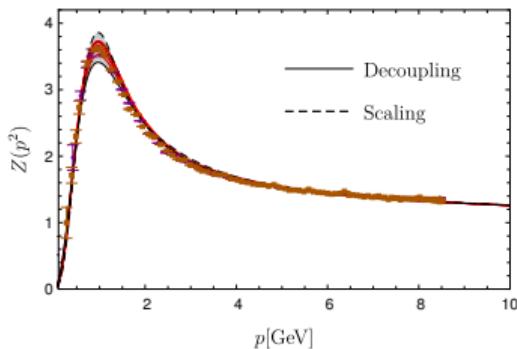
Start with **pure gauge theory**.

# Landau gauge propagators

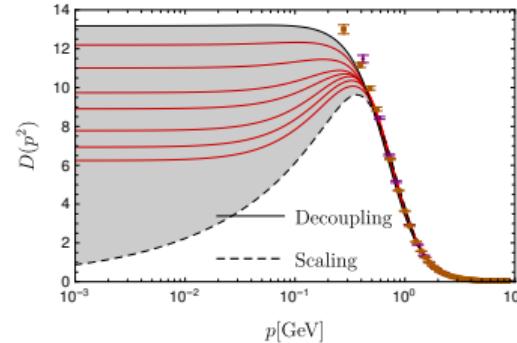
Self-contained: Only external input is the coupling!

[MQH, Phys.Rev.D 101 (2020)]

Gluon dressing function:

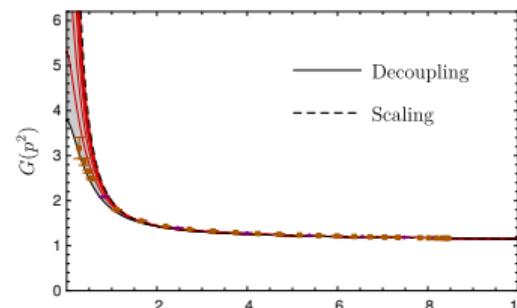


Gluon propagator:



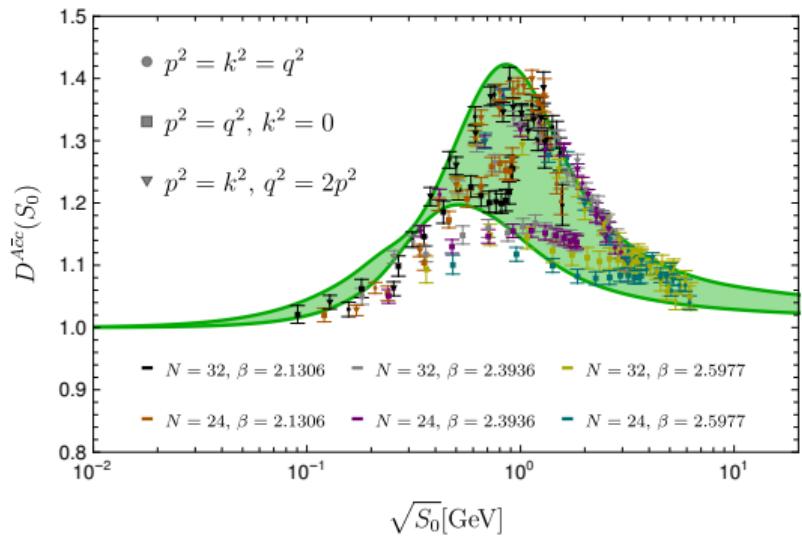
- Family of solutions: Nonperturbative completions of Landau gauge [Maas, Phys. Lett. B 689 (2010)]?
- Realized by condition on  $G(0)$  [Fischer, Maas, Pawłowski, Annals Phys. 324 (2008); Alkofer, MQH, Schwenzer, Phys. Rev. D 81 (2010)]

Ghost dressing function:



# Ghost-gluon vertex

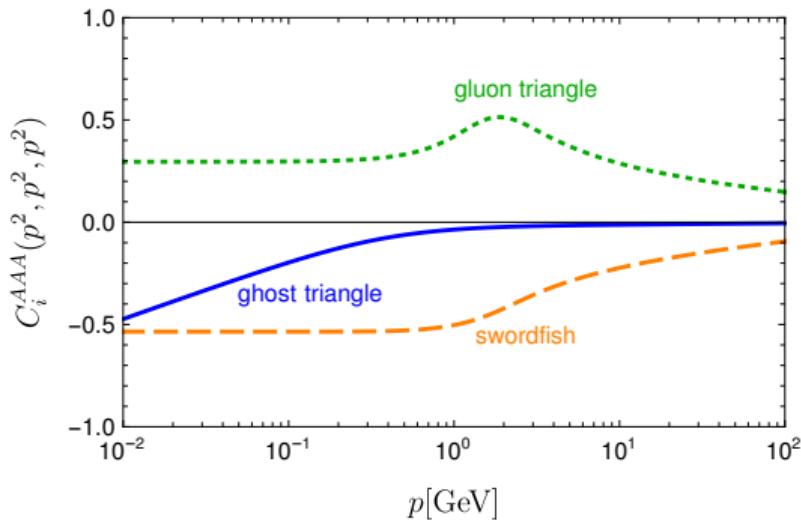
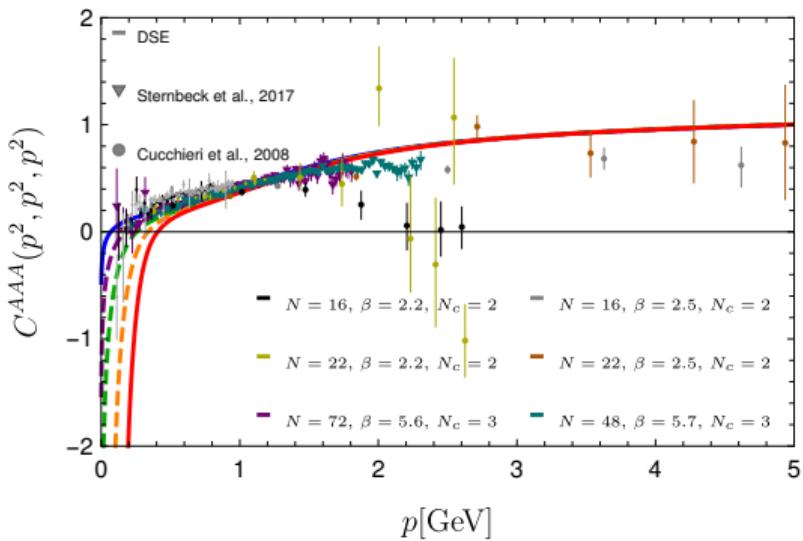
Ghost-gluon vertex:



[Maas, SciPost Phys. 8 (2019);  
MQH, Phys. Rev. D 101 (2020)]

- Nontrivial kinematic dependence of ghost-gluon vertex
- Qualitative agreement with lattice results, though some quantitative differences.

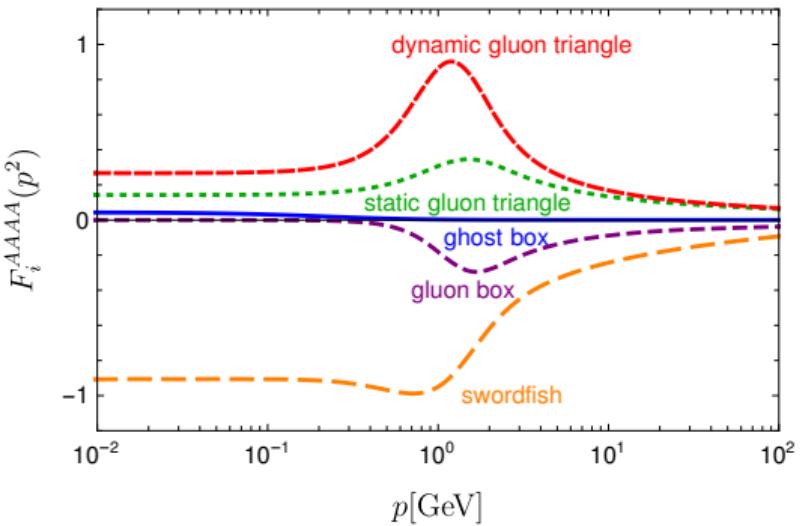
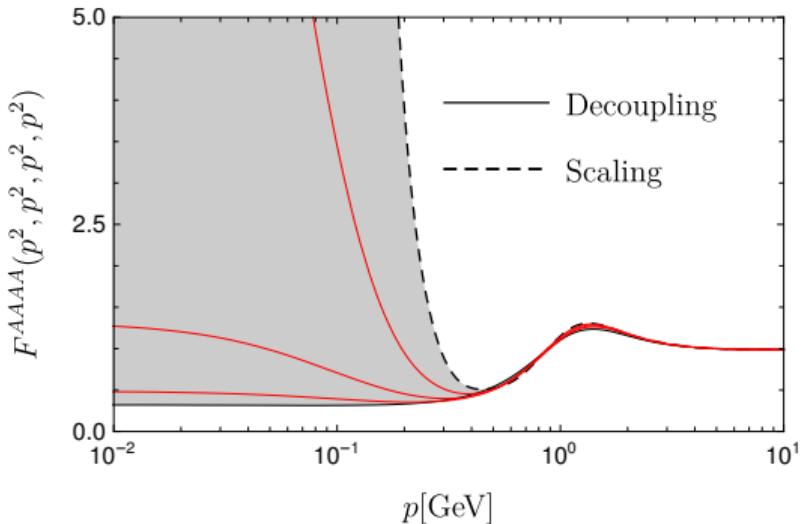
# Three-gluon vertex



- Simple kinematic dependence of three-gluon vertex (only singlet variable of  $S_3$ )
- Large cancellations between diagrams

[Cucchieri, Maas, Mendes, Phys. Rev. D 77 (2008); Sternbeck et al., 1702.00612; MQH, Phys. Rev. D 101 (2020)]

# Four-gluon vertex



- Close to tree-level
- Large cancellations between diagrams

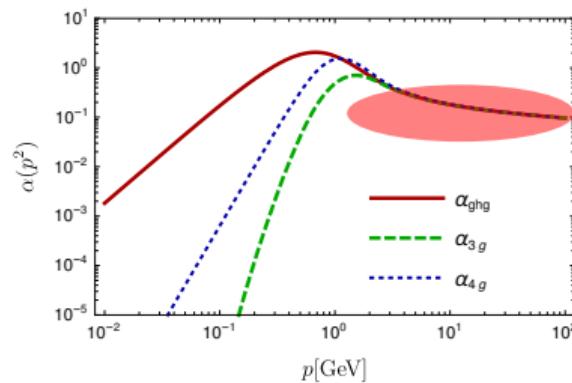
[MQH , Phys. Rev. D 101 (2020)]

# Gauge invariance

[MQH, Phys. Rev. D 101 (2020)]

Couplings can be extracted from each vertex.

- Slavnov-Taylor identities (gauge invariance): Agreement perturbatively (UV) necessary.
- Difficult to realize: Small deviations → Couplings cross and do not agree.
- Here: Vertex couplings agree down to GeV regime (IR can be different).

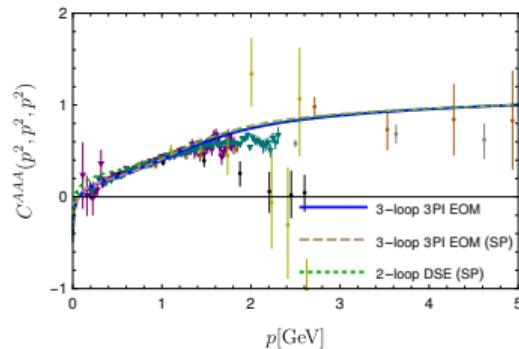


# Stability of the solution

- Agreement with lattice results.

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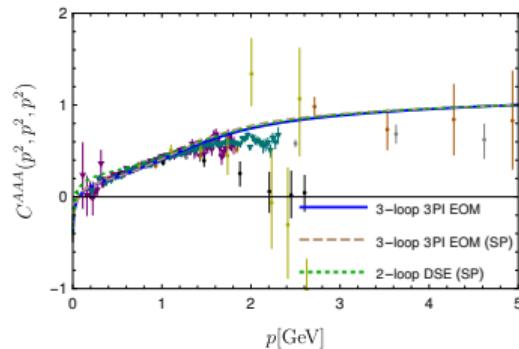
- Agreement with lattice results.
- Concurrence between functional methods:  
3PI vs. 2-loop DSE:



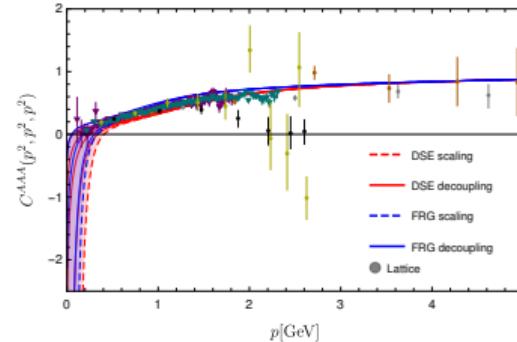
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3PI vs. 2-loop DSE:



DSE vs. FRG:

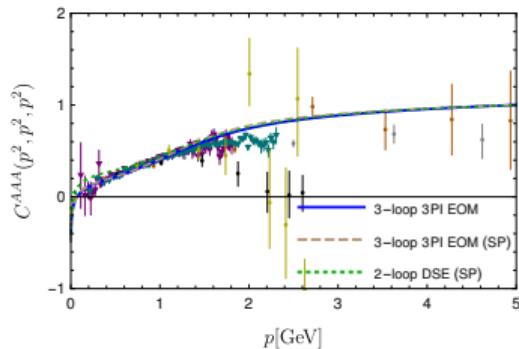


[Cucchieri, Maas, Mendes, Phys.Rev.D77 (2008); Sternbeck et al., Proc.Sci. LATTICE2016 (2017); Cyrol et al., Phys.Rev.D 94 (2016); MQH, Phys.Ref.D101 (2020)]

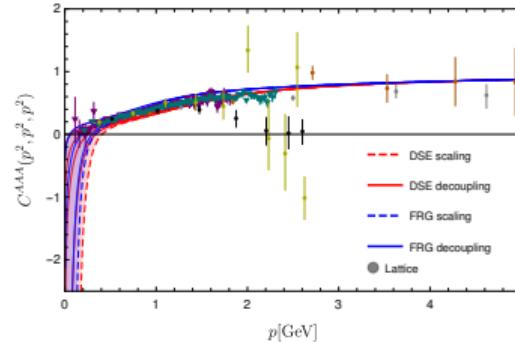
# Stability of the solution

- Agreement with lattice results.
- Concurrence between functional methods:

3PI vs. 2-loop DSE:



DSE vs. FRG:



[Cucchieri, Maas, Mendes, Phys.Rev.D77 (2008); Sternbeck et al., Proc.Sci. LATTICE2016 (2017); Cyrol et al., Phys.Rev.D 94 (2016); MQH, Phys.Rev.D101 (2020)]

- Stable against extensions: Four-point functions

[MQH, Phys.Rev.D93 (2016); MQH, Eur.Phys.J.C 77 (2017); Corell, SciPost Phys. 5 (2018); MQH, Phys.Rept. 879 (2020)]

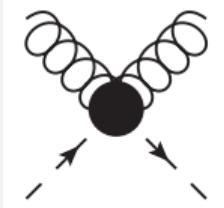
# The two-ghost-two-gluon vertex

Non-primitively divergent correlation function → No guide from tree-level tensor. → Use full basis.

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## Two-ghost-two-gluon vertex



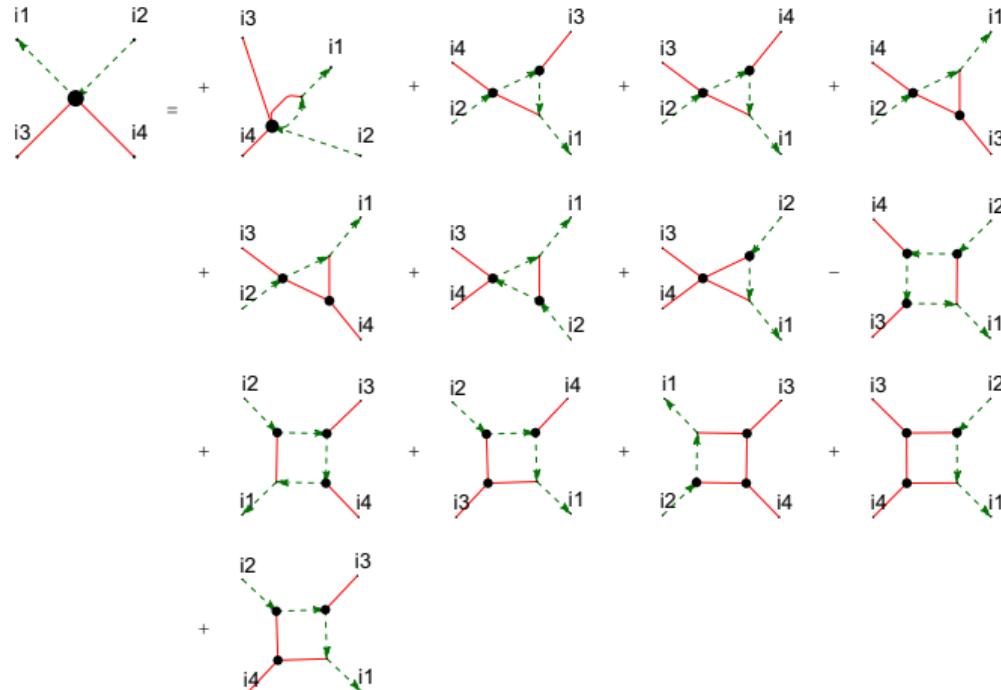
$$\Gamma_{\mu\nu}^{AA\bar{c}c,abcd}(p, q; r, s) = \textcolor{red}{g^4} \sum_{k=1}^{40} \rho_{\mu\nu}^{k,abcd} D_{k(i,j)}^{AA\bar{c}c}(p, q; r, s)$$

with

$$\rho_{\mu\nu}^{k,abcd} = \sigma_i^{abcd} \tau_{\mu\nu}^j, \quad k = k(i,j) = 5(i-1) + j$$

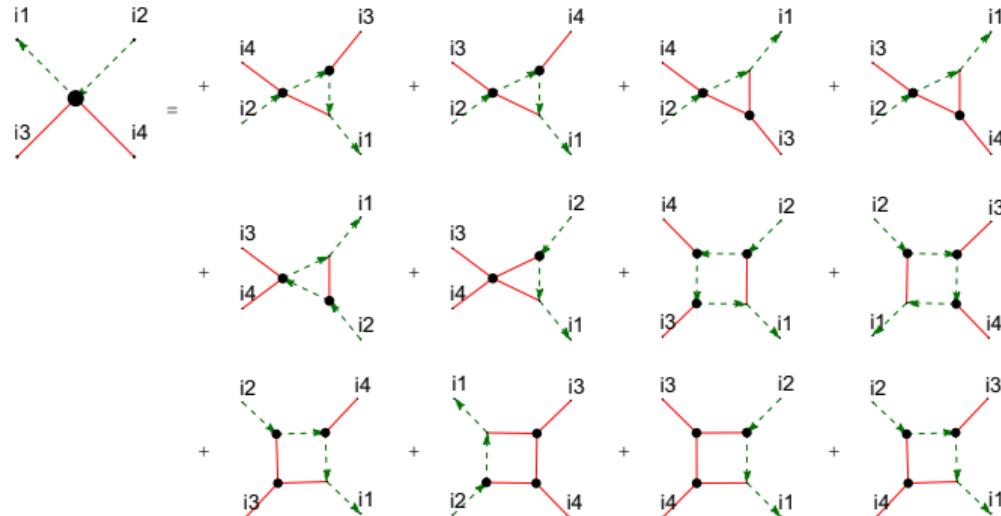
# The two-ghost-two-gluon vertex DSE

2 DSEs, choose the one with the ghost leg attached to the bare vertex  
 → Truncation discards only one diagram.



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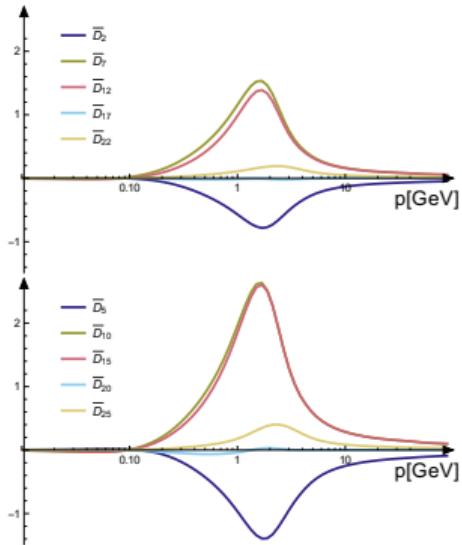
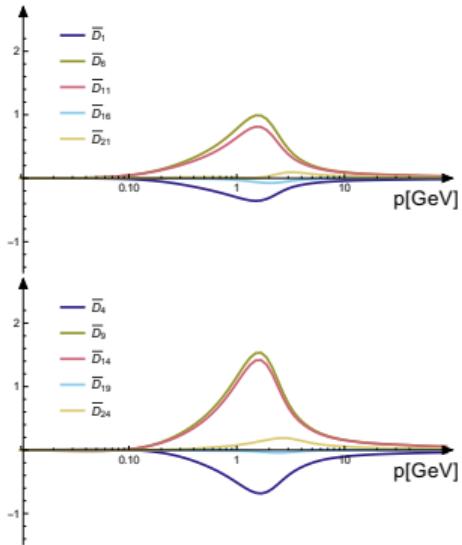
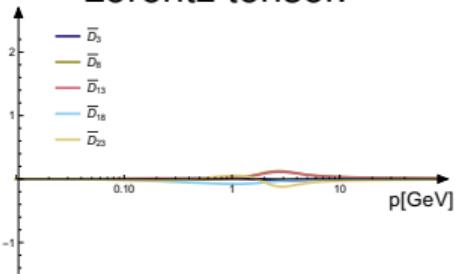
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# Results for the two-ghost-two-gluon vertex

Kinematic approximation: one-momentum configuration

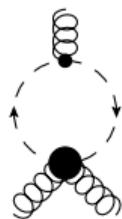
- Dimensionless dressing functions  $\bar{D}_k$ .
- Each plot one Lorentz tensor.



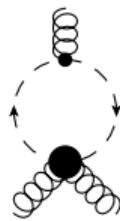
- Two classes of dressings: **13 very small**, **12 not small**
- No nonzero solution for  $\{\sigma_6, \sigma_7, \sigma_8\}$  found.

[MQH, Eur.Phys.J.C 77 (2017)]

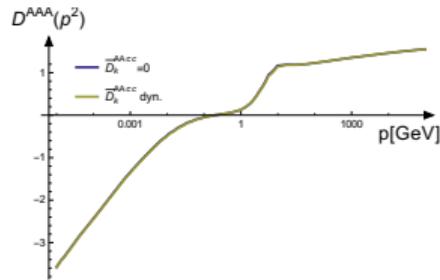
# Influence of two-ghost-two-gluon vertex



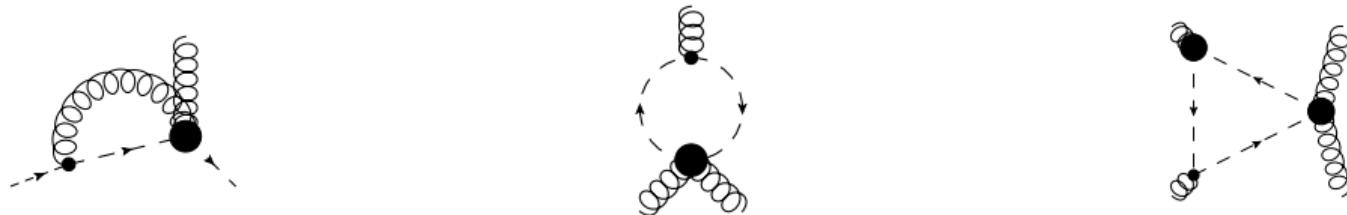
# Influence of two-ghost-two-gluon vertex



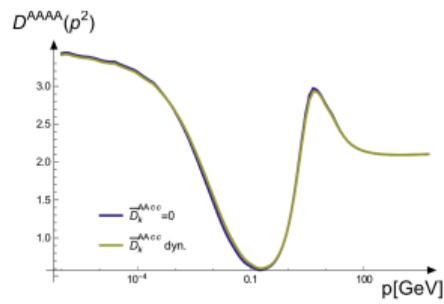
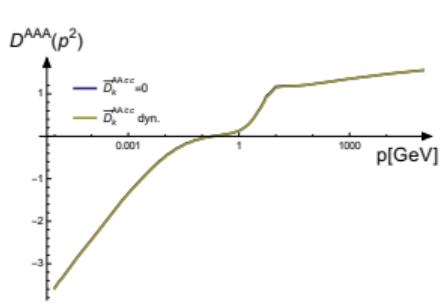
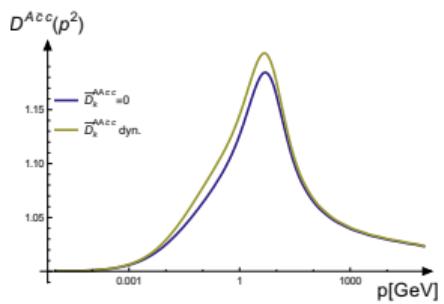
Coupled system of ghost-gluon, three-gluon and four-gluon vertices **with and without** two-ghost-two-gluon vertex [MQH, Eur.Phys.J.C 77 (2017)]:



# Influence of two-ghost-two-gluon vertex



Coupled system of ghost-gluon, three-gluon and four-gluon vertices **with and without** two-ghost-two-gluon vertex [MQH, Eur.Phys.J.C 77 (2017)]:



- Color structure: only small dressings in the three-gluon DSE → no change.
- **Small** influence on ghost-gluon vertex (< 1.7%)

# Gauges

Choose a convenient gauge!

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Landau gauge

- Manifest covariance
- Exists for continuum and lattice methods. → Comparisons
- Transverse gluon propagator → transverse sector independent of longitudinal one
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- Coulomb gauge
- Maximally Abelian gauge
- Linear covariant gauge
- ...
- Interpolating gauges!

# Linear covariant gauge

- The Landau gauge is the endpoint ( $\xi = 0$ ) of linear covariant gauges.

- $\mathcal{L}_{\text{gf}} = \frac{1}{2\xi}(\partial \cdot A)^2 - \bar{c} M c$

- Gluon propagator:  $D(p^2) = \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \frac{Z(p^2)}{p^2} + \xi \frac{p_\mu p_\nu}{p^4}$

## Lattice and functional investigations:

[Aguilar, Binosi, J. Papavassiliou, Phys. Rev. D91 (2015); Bicudo et al., Phys. Rev. D92 (2015); MQH, Phys. Rev. D91 (2015); Aguilar, Binosi, Papavassiliou, Phys. Rev. D95 (2017); Cucchieri et al., '18; Napetschnig, Alkofer, MQH, Pawłowski, Phys. Rev. D 104 (2021); ...]

# Nielsen identities

- Describe gauge parameter dependence of correlation functions by a differential equation.

$$\frac{\partial \Gamma}{\partial \xi} \Big|_{\chi=0} = \int dx \left( \frac{\partial \delta \Gamma}{\partial \chi \delta A_\mu^a} \frac{\delta \Gamma}{\delta A_\mu^{*a}} + \frac{\delta \Gamma}{\delta A_\mu^a} \frac{\partial \delta \Gamma}{\partial \chi \delta A_\mu^{*a}} + \frac{\partial \delta \Gamma}{\partial \chi \delta c^a} \frac{\delta \Gamma}{\delta c^{*a}} - \frac{\delta \Gamma}{\delta c^a} \frac{\partial \delta \Gamma}{\partial \chi \delta c^{*a}} + i b^a \frac{\partial \delta \Gamma}{\partial \chi \delta \bar{c}^a} \right) \Big|_{\chi=0}.$$

- Traditional use: Show gauge parameter independence of pole masses.

# Nielsen identities

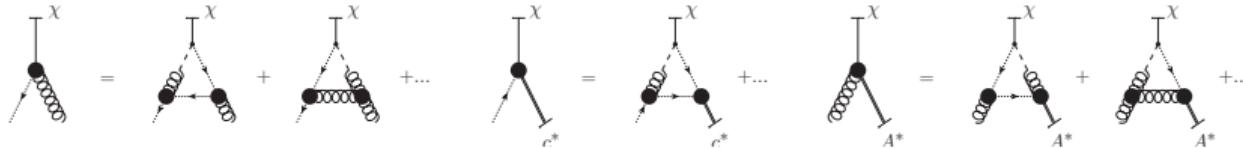
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- Traditional use: Show gauge parameter independence of pole masses.
- Here: Solve them for the propagators

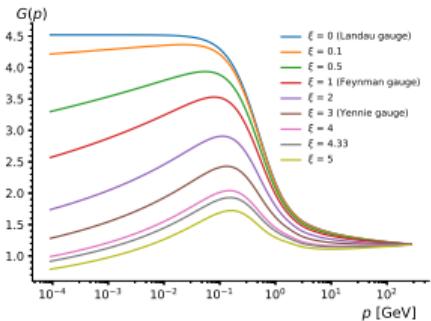
$$\partial_\xi Z(p^2; \xi) = K_Z(p^2; \xi) Z(p^2; \xi), \quad \partial_\xi G(p^2; \xi) = K_G(p^2; \xi) G(p^2; \xi)$$

- Initial condition: Landau gauge ( $\xi = 0$ )
- $K_Z, K_G$ : nonperturbative one-loop integrals



# Ghost propagator

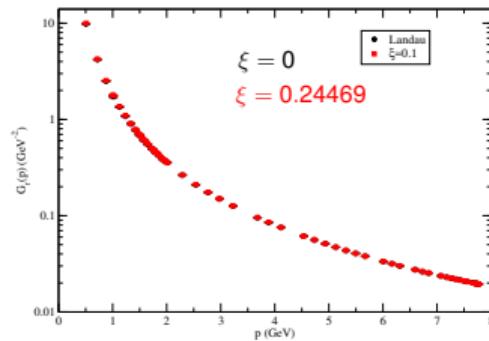
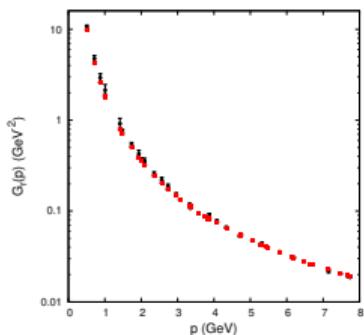
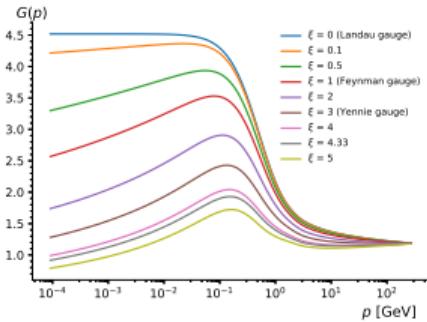
[Napetschnig, Alkofer, MQH, Pawłowski, Phys.Rev.D 104 (2021)]



- Logarithmic IR suppression for  $\xi > 0$   
[Aguilar, Binosi, J. Papavassiliou, Phys.Rev. D91 (2015); MQH, Phys. Rev. D91 (2015)]
- Otherwise effects small for low  $\xi$ .

# Ghost propagator

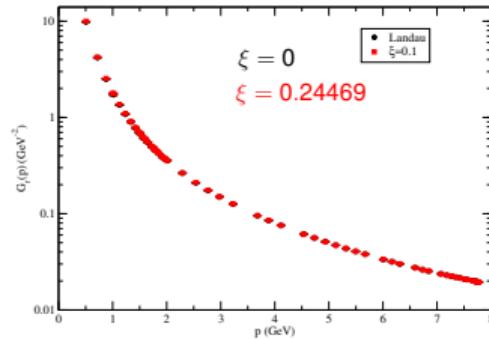
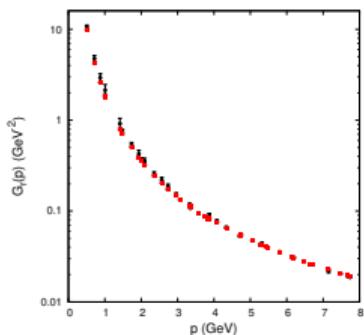
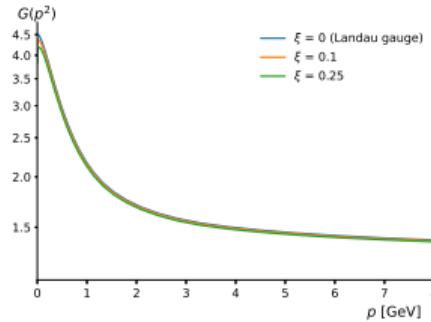
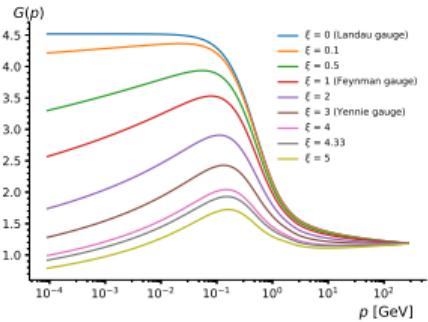
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[Cucchieri et al. '18]

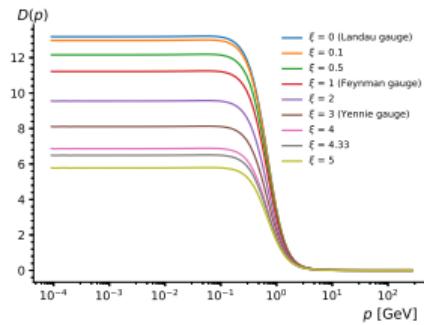
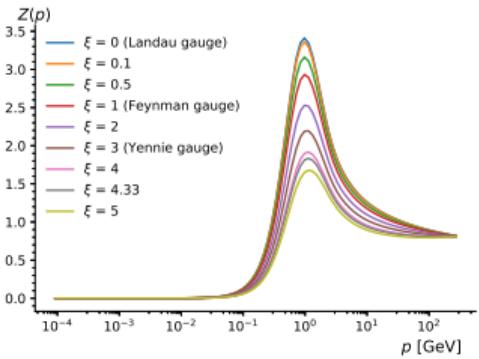
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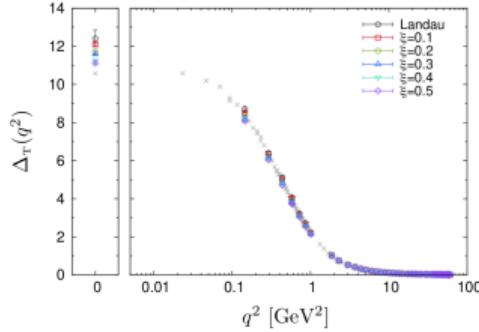
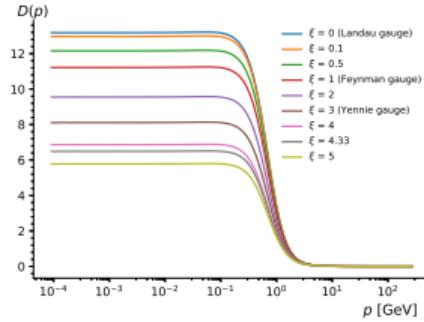
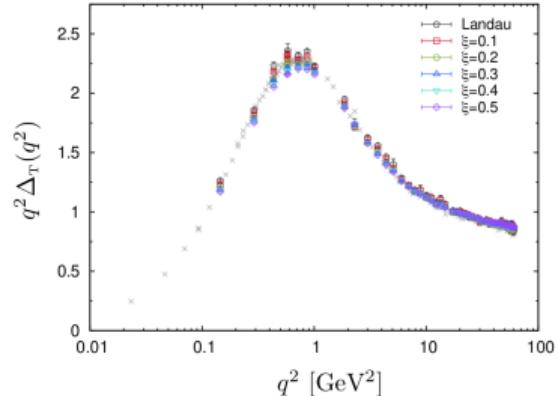
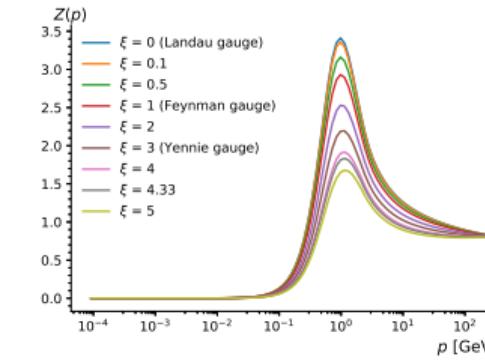
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# Gluon propagator



[Napetschnig, Alkofer, MQH,  
Pawlowski, Phys.Rev.D 104 (2021)]

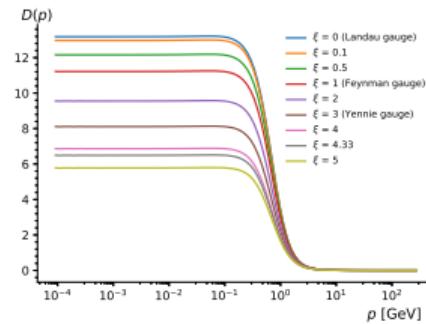
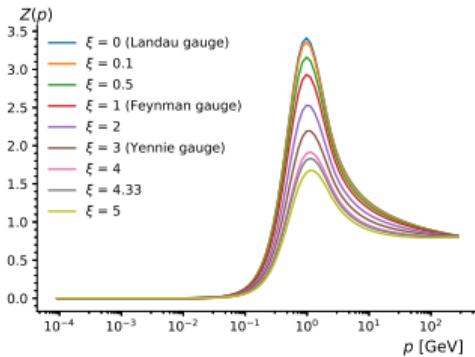
# Gluon propagator



[Napetschnig, Alkofer, MQH,  
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[Bicudo et al., Phys. Rev. D92  
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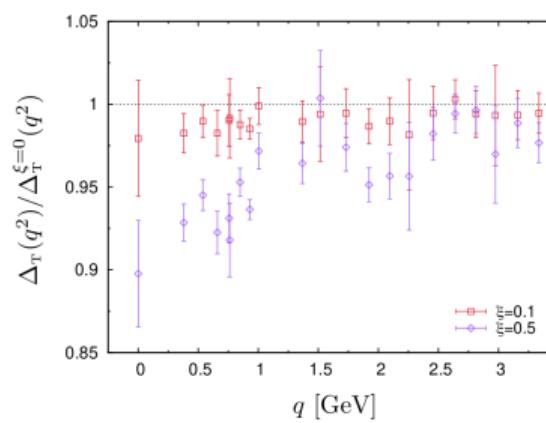
Ratios from Nielsen identities:

$\xi = 0.1$ :

- 0 GeV: 0.98
- 1 GeV: 0.98

$\xi = 0.5$ :

- 0 GeV: 0.92
- 1 GeV: 0.93



[Bicudo et al., Phys. Rev. D92  
(2015)]

# Stability of solving Nielsen identities

Nontrivial check: UV behavior

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Exceptional values of  $\xi$  where 1-loop  
**anomalous dimensions vanish**

$$Z_{\text{UV}}(p^2) = Z(s) \left(1 + \omega(s) \ln \frac{p^2}{s}\right)^{-\frac{13-3\xi}{22}}$$

$$G_{\text{UV}}(p^2) = G(s) \left(1 + \omega(s) \ln \frac{p^2}{s}\right)^{-\frac{9-3\xi}{44}}$$

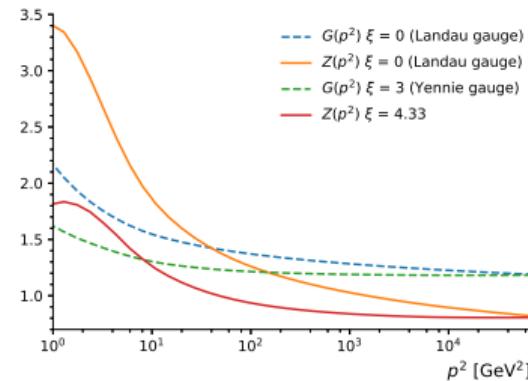
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→ Exceptionally stable process.

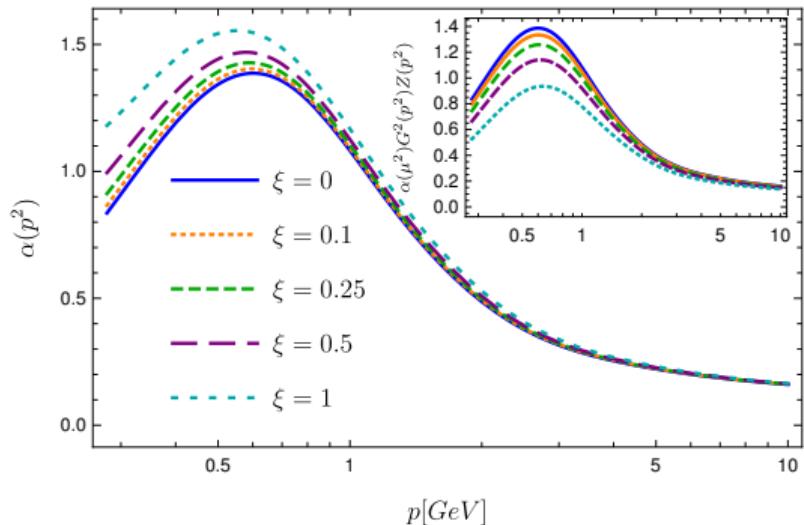
Remaining uncertainties: vertices

[Napetschnig, Alkofer, MQH, Pawłowski, Phys.Rev.D 104 (2021)]

# Coupling

Only leading behavior of coupling is gauge independent (first coefficient of  $\beta$  function).

Coupling:



- Universality at high momentum recovered.
- Deviations in midmomentum small.

[Napetschnig, Alkofer, MQH, Pawłowski, Phys.Rev.D 104 (2021)]

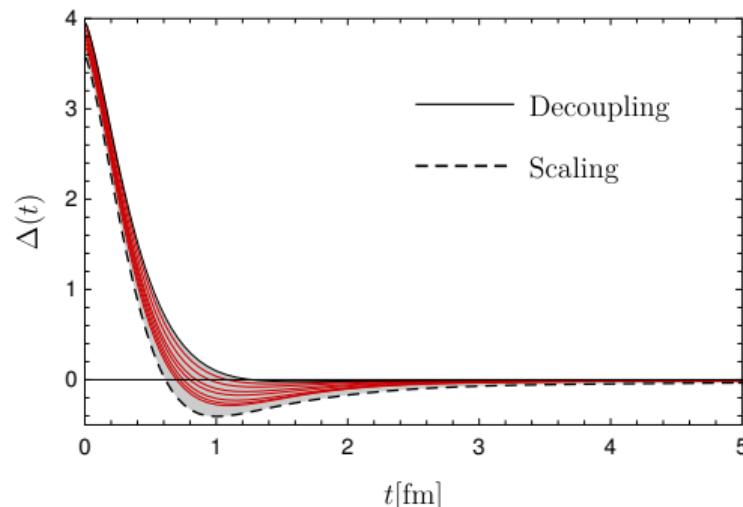
# Schwinger function

$$\Delta(t) = D(t, \vec{p} = \vec{0}) = \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} e^{-ip_0 t} D(p_0, \vec{0}) = \frac{1}{\pi} \int_0^{\infty} dp_0 \cos(p_0 t) D(p_0, \vec{0})$$

Positivity violation in spectral density if  
 $\Delta(t) < 0$  for any  $t$ :

$$\Delta(t) = \int_0^{\infty} d\omega e^{-\omega t} \rho(\omega^2)$$

[MQH, Phys.Rev.D 101 (2020)]



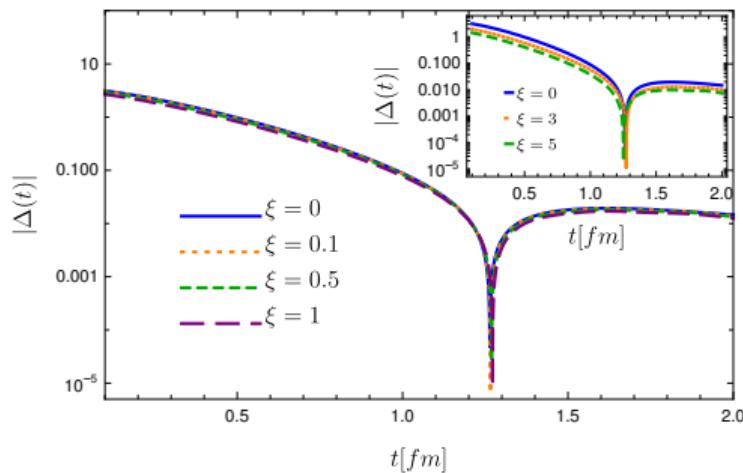
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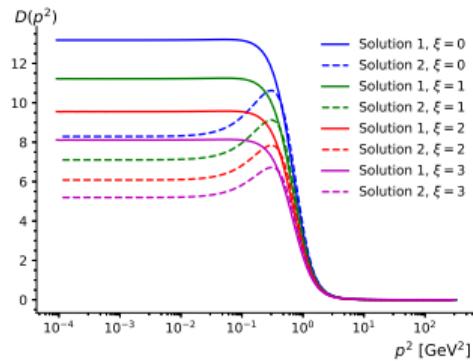
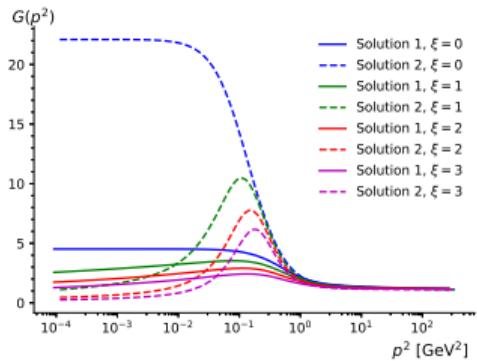


# Family of solutions at $\xi > 0$

Various solutions in Landau gauge. What happens at  $\xi > 0$ ?

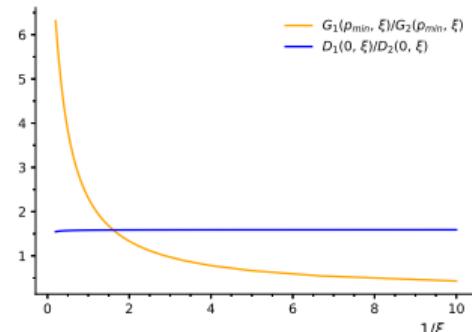
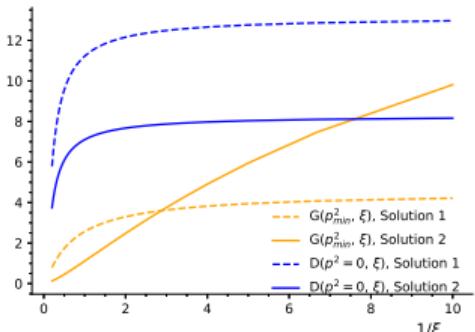
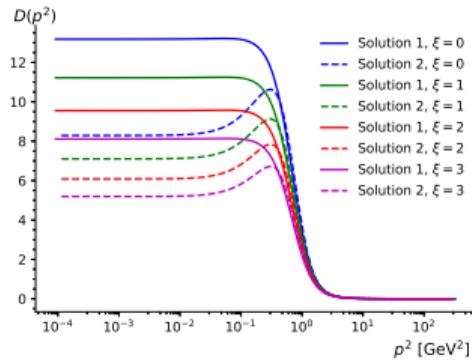
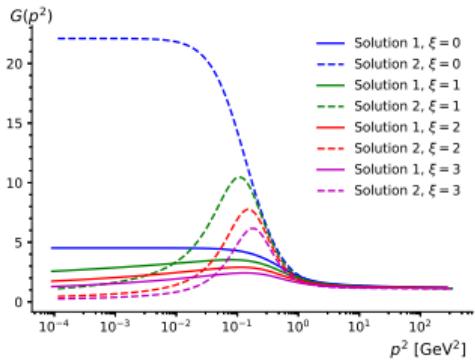
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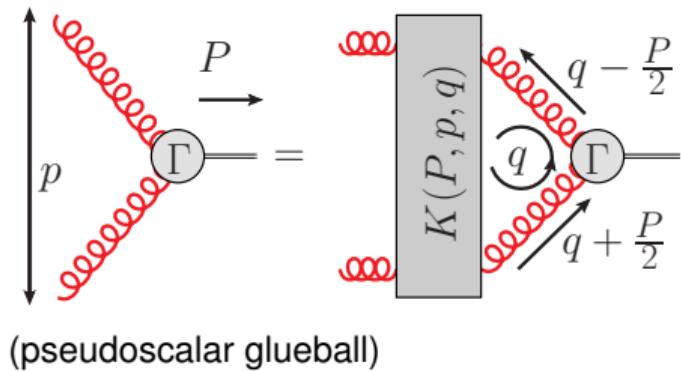
Various solutions in Landau gauge. What happens at  $\xi > 0$ ?



- $\xi \rightarrow \infty$  Gauge fixing washed out.
- Ghost dressings approach each other.

[Napetschnig, Alkofer, MQH,  
Pawlowski, Phys.Rev.D 104 (2021)]

# Solving a bound state equation

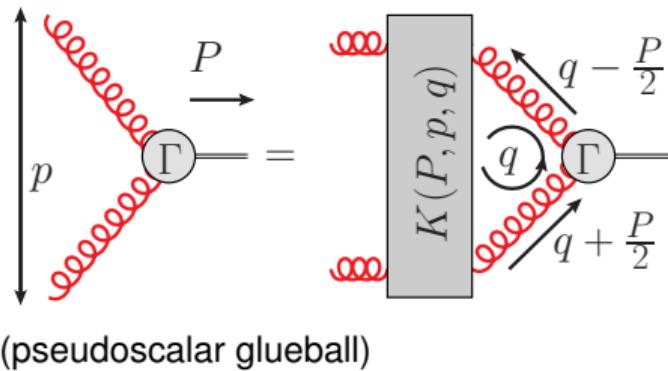


$$\Gamma(P) = \mathcal{K} \cdot \Gamma(P)$$

→ Eigenvalue problem for  $\Gamma(P)$ :

- ① Solve for  $\lambda(P)$ .
- ② Find  $P$  with  $\lambda(P) = 1$ .  
 $\Rightarrow M^2 = -P^2$

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⇒  $M^2 = -P^2$

However:

Propagators are probed at  $\left(q \pm \frac{P}{2}\right)^2 = \frac{P^2}{4} + q^2 \pm \sqrt{P^2 q^2} \cos \theta = -\frac{M^2}{4} + q^2 \pm i M \sqrt{q^2} \cos \theta$   
 → Complex for  $P^2 < 0$ !

Time-like quantities ( $P^2 < 0$ ) → Correlation functions for complex arguments.

# Correlation functions in the complex plane

Standard integration techniques fail. 

$$\int d^4 q \rightarrow \int_{\Lambda_{IR}^2}^{\Lambda_{UV}^2} dq^2 \int d\theta_1$$

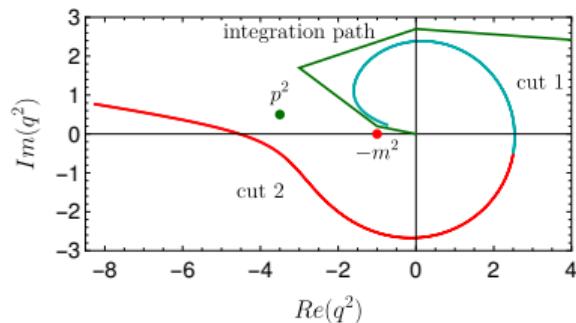
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→ Adapted technique: Contour deformation  
(QED: [Maris, Phys.Rev.D52, (1995)]).



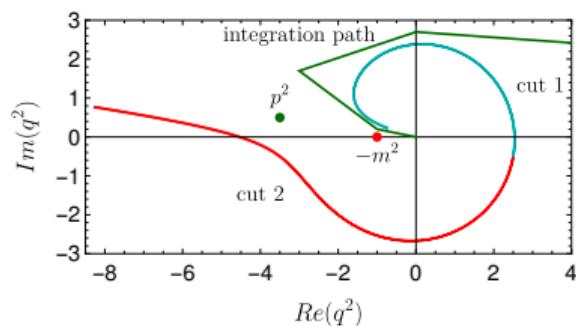
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Recent resurgence, e.g.: [Alkofer et al., Phys.Rev.D 70 (2004); Strauss, Fischer, Kellermann, Phys.Rev.Lett. 109 (2012); Windisch, MQH, Alkofer, Phys.Rev.D 87 (2013), Acta Phys.Polon.Supp. 6 (2013); Weil, Eichmann, Fischer, Williams, Phys.Rev.D 96 (2017); Williams, Phys.Lett.B 798 (2019); Miramontes, Sanchis-Alepuz, Eur.Phys.J.A 55 (2019); Eichmann, Duarte, Pena Stadler, Phys.Rev.D 100 (2019); Fischer, MQH, Phys.Rev.D 102 (2020); Eichmann, Ferreira, Stadler, Phys.Rev.D 105 (2022); Miramontes, Sanchis-Alepuz, Phys.Rev.D 103 (2021); Miramontes, Alkofer, Fischer, Sanchis-Alepuz, '22; ...]

# Landau gauge propagators in the complex plane

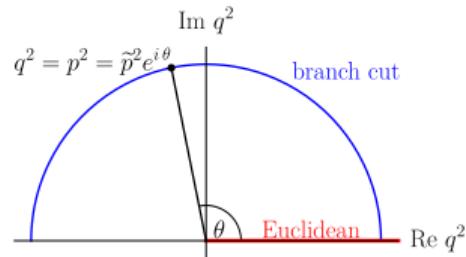
Simpler truncation:

$$\text{Diagram A}^{-1} = \text{Diagram B}^{-1} - \frac{1}{2} \text{Diagram C} + \text{Diagram D}$$

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$$\text{propagator}^{-1} = \text{propagator}^{-1} - \frac{1}{2} \text{loop} + \text{loop}$$

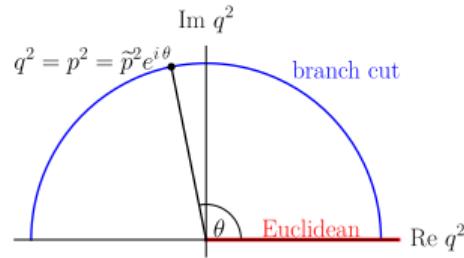


→ Opening at  $q^2 = p^2$ .

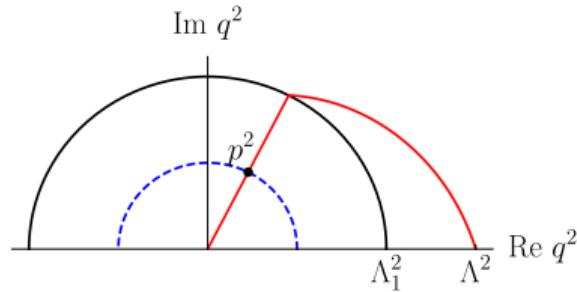
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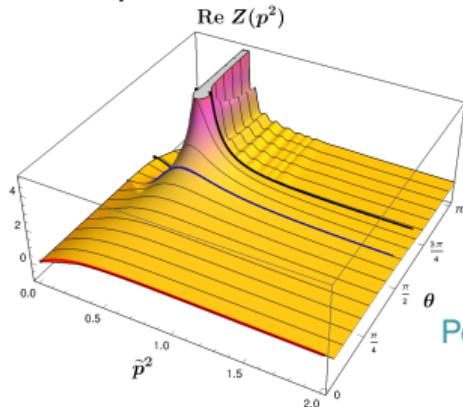
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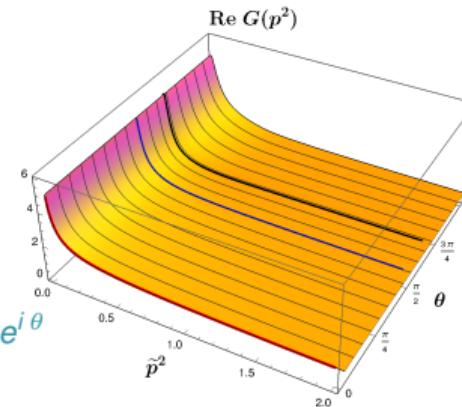
Appearance of branch cuts for complex momenta forbids integration directly to cutoff.

# Landau gauge propagators in the complex plane

Ray technique for self-consistent solution of a DSE:



$$\text{Polar coordinates: } p^2 = \tilde{p}^2 e^{i\theta}$$



- Current truncation leads to a pole-like structure in the gluon propagator.
- Analyticity up to 'pole' confirmed by various tests (Cauchy-Riemann, Schlessinger, reconstruction)

[Fischer, MQH, Phys.Rev.D 102 (2020)]

# Extrapolation of $\lambda(P^2)$

## Extrapolation method

- Extrapolation to time-like  $P^2$  using Schlessinger's continued fraction method (proven superior to default Padé approximants) [Schlessinger, Phys.Rev.167 (1968)]
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$$f(x) = \frac{f(x_1)}{1 + \frac{a_1(x-x_1)}{1 + \frac{a_2(x-x_2)}{1 + \frac{a_3(x-x_3)}{\dots}}}}$$

Coefficients  $a_i$  can be determined such that  $f(x)$  exact at  $x_i$ .

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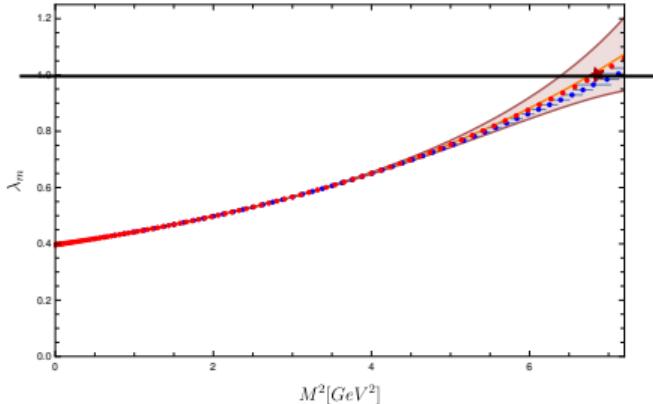
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**Test extrapolation for solvable system:**  
**Heavy meson** [MQH, Sanchis-Alepuz, Fischer, Eur.Phys.J.C 80 (2020)]

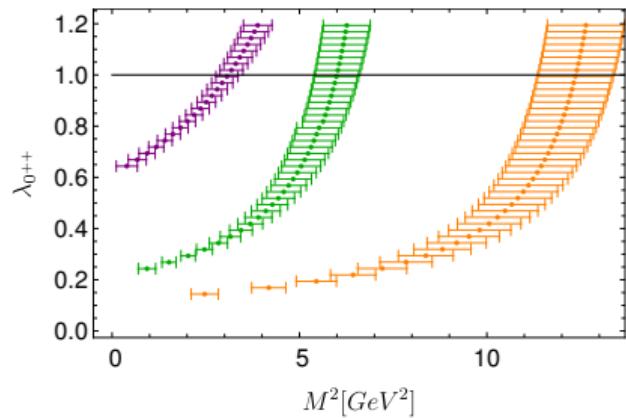
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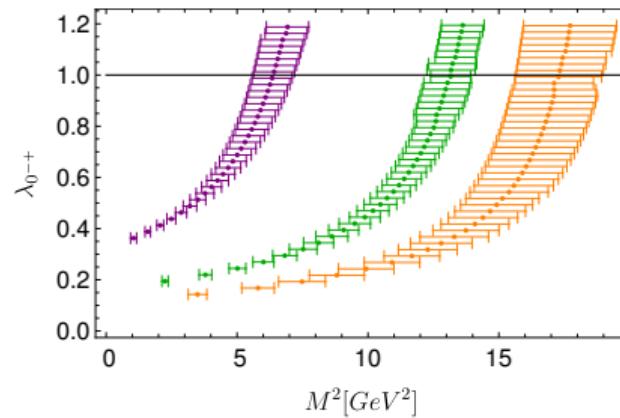


# Extrapolation for glueball eigenvalue curves

Scalar glueball ( $0^{++}$ ):

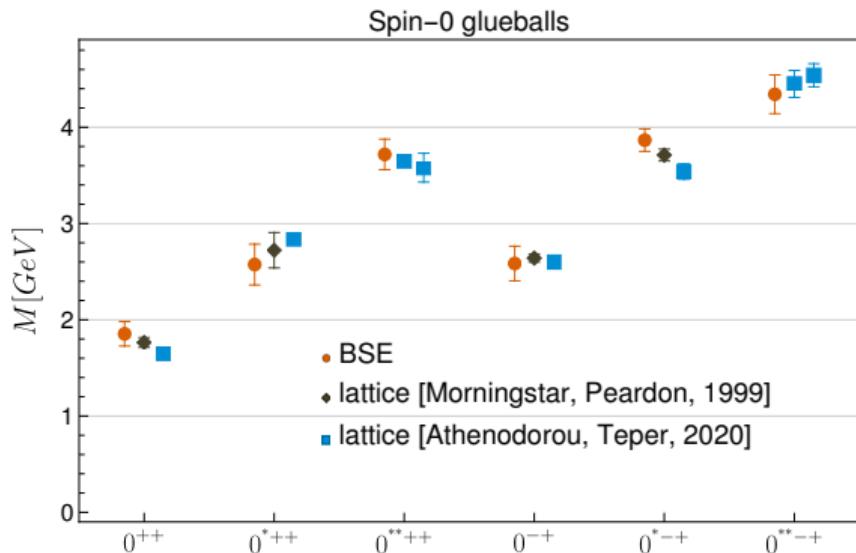


Pseudoscalar glueball ( $0^{-+}$ ):



Several curves: ground state and excited states.

# Glueball results J=0

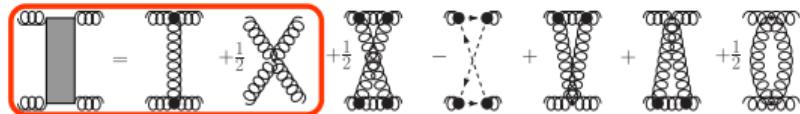


Lattice  $0^{**\pm+}$ : Conjectured based on irred. rep. of octahedral group

All results for  
 $r_0 = 1/418(5)$  MeV.

Spectrum independent of  $G(0)!$   $\rightarrow$  Family of solutions yields the same physics.

# Higher order diagrams



One-loop diagrams only:

[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C80 (2020); MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C81 (2021)]

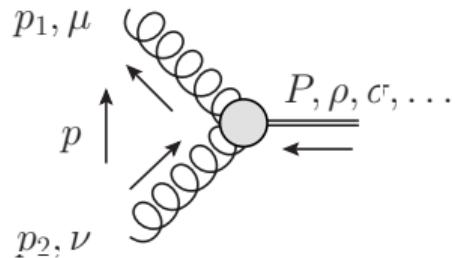
$J = 0$  and subleading kernel contributions:

Subleading effects: none ( $0^{-+}$ ), tiny ( $< 2\%$ ,  $0^{++}$ ):

[MQH, Fischer, Sanchis-Alepuz, EPJ Web Conf. 258 (2022); MQH, Fischer, Sanchis-Alepuz, HADRON2021, arXiv:2201.05163]

# Glueball amplitudes

$$\Gamma_{\mu\nu\rho\sigma\dots}(p_1, p_2) = \sum \tau^i_{\mu\nu\rho\sigma\dots}(p_1, p_2) h_i(p_1, p_2)$$

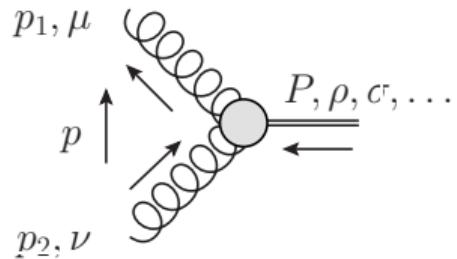


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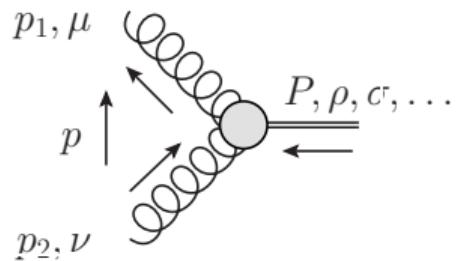


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Symmetric, traceless tensor  
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Enforce with **spin projectors**.  
 $\rightarrow$  10 tensors

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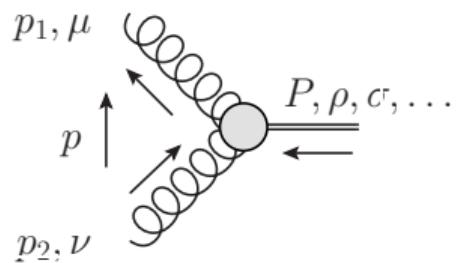


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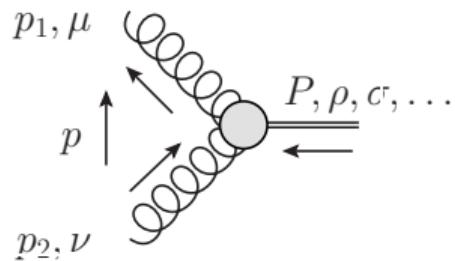


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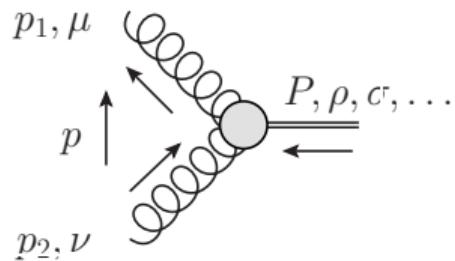
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$J$	$P = +$	$P = -$
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'Low' number of tensors, but high-dimensional tensors!

$\rightarrow$  Computational cost increases with  $J$ .

# $J = 1$ glueballs

Landau-Yang theorem

Two-photon states cannot couple to  $J^P = \mathbf{1}^\pm$  or  $(2n+1)^-$

[Landau, Dokl.Akad.Nauk SSSR 60 (1948); Yang, Phys. Rev. 77 (1950)].

(→ Exclusion of  $J = 1$  for Higgs because of  $h \rightarrow \gamma\gamma$ .)

Applicable to glueballs?

- Not in this framework, since gluons are not on-shell.
- Presence of  $J = 1$  states is a dynamical question.

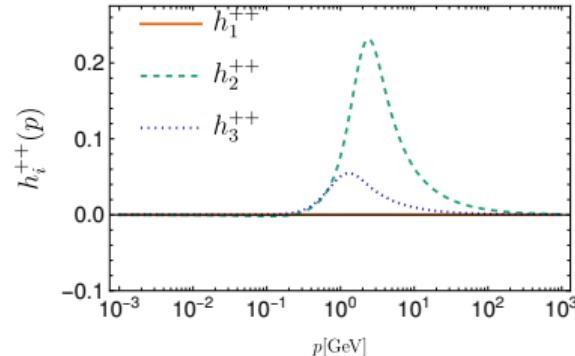
$J = 1$  not found here.

# Amplitudes

Information about significance of single parts.

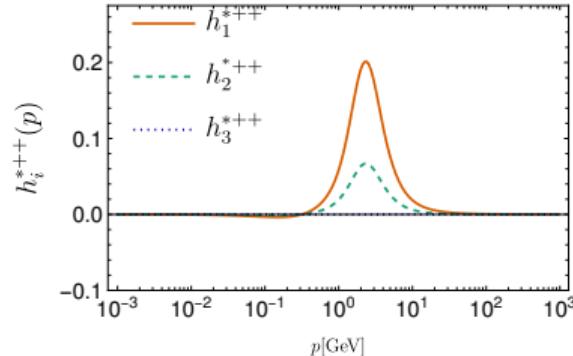
Ground state scalar glueball:

Amplitudes  $0^{++}$



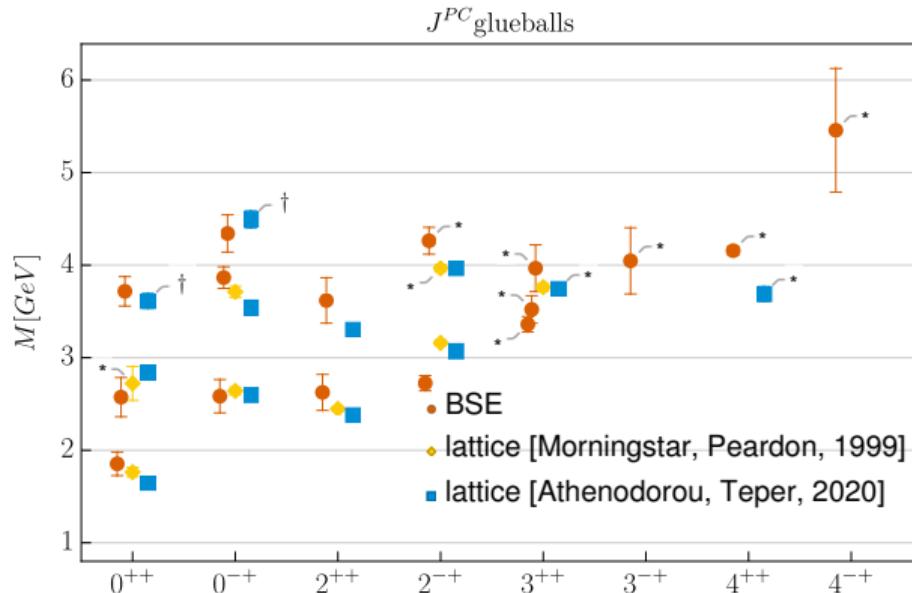
Excited scalar glueball:

Amplitudes  $0^{*++}$



- Amplitudes have different behavior for ground state and excited state. Useful guide for future developments.
- Meson/glueball amplitudes: **Information about mixing.**

# Glueball results



Lattice:

\*: identification with some uncertainty

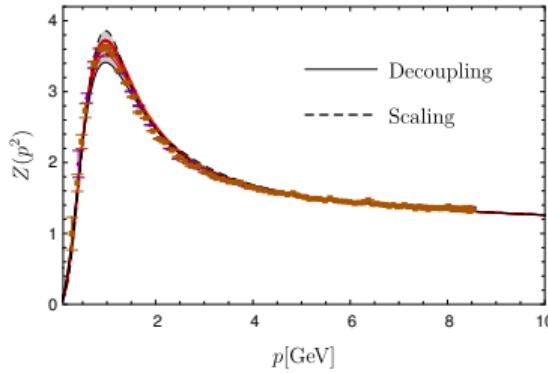
†: conjecture based on irred. rep of octahedral group

[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C81 (2021)]

- Agreement with lattice results
- (New states:  $0^{***++}$ ,  $0^{***-+}$ ,  $3^{-+}$ ,  $4^{-+}$ )

# Summary

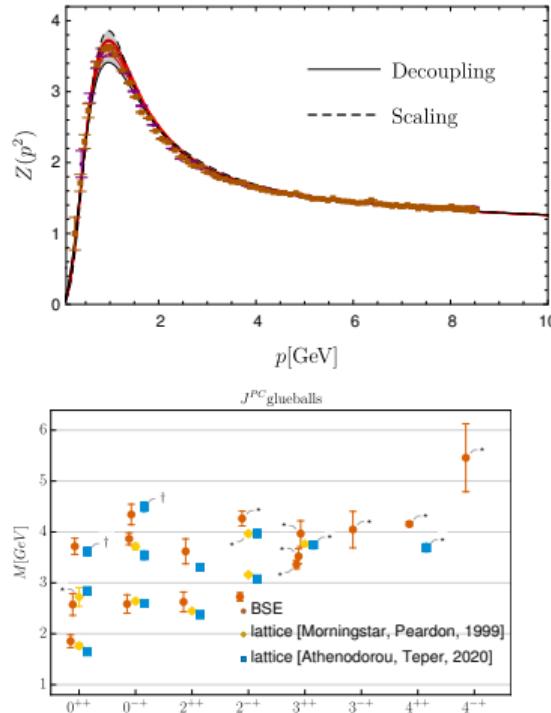
- Alternative to models in functional equations: **Direct calculation** of input for bound state equations.
- Large system of equations may be necessary.
- Independent tests:
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lattice + continuum
  - Extensions



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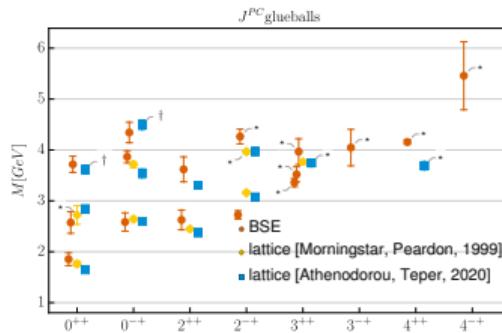
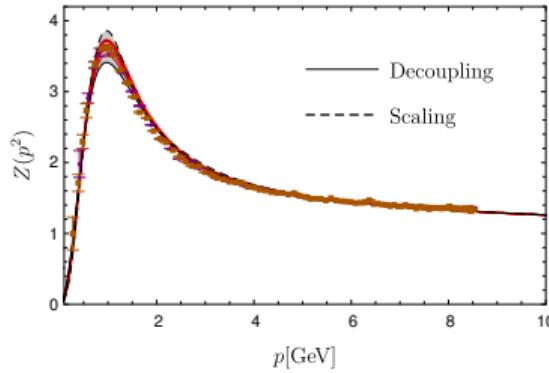
Spectrum from **first principles** for pure glueballs.



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Thank you for your attention.